1. Write each of the following complex numbers in the form $x + iy$ with $x, y \in \mathbb{R}$ and $i^2 = -1$. [1pt each]

(a) $\frac{2 - 3i}{3 + 4i}$,

(b) $\frac{1}{1 - i} + \frac{i}{1 - 2i} + \frac{1}{3 - 2i}$.

2. [1pt each]

(a) Express $-\sqrt{3} + i$ in the polar form.

(b) Find all the cube roots of $8i$.

(c) Solve $z^2 = 11 + 60i$.

3. [2pts] Put $w = \frac{i-1}{1+z}$. Prove that $|z| < 1$ if and only if $\text{Im}(w) > 0$.

4. [2pts each]

At right is a picture of the lines $\text{Re}(z) \in \{-2, -1, 0, 1, 2\}$ and $\text{Im}(z) \in \{-2, -1, 0, 1, 2\}$ in $\mathbb{C}$, with with the rectangle $\{z \in \mathbb{C} \mid 0 \leq \text{Re}(z) \leq 1, 1 \leq \text{Im}(z) \leq 2\}$ shaded in.

Draw what happens to this picture under the maps

(a) $f(z) = z^2$,

(b) $f(z) = \frac{1}{z}$.

5. [2pts each] The purpose of this problem is to prove that the image of a circle under a Möbius transformation is always either a line or a circle.

Suppose that $f(x) = \frac{az + b}{cz + d}$ is a Möbius transformation (i.e., $a, b, c, d \in \mathbb{C}, ad - bc \neq 0$).

(a) Explain how you know that $f$ is invertible, and give a formula for the inverse function $f^{-1}(z)$.
(b) Suppose that $p \in \mathbb{C}$, $r \in \mathbb{R}$ (with $r > 0$), and let $C$ be the circle $C = \left\{ z \mid |z - p| = r \right\}$. (i.e., $C$ is the circle of radius $r$ centred at $p$).

Explain why the image of $C$ under the map $f$ is the same as the set $\left\{ z \mid |f^{-1}(z) - p| = r \right\}$.

(c) Use part (b) and a result from class to show that the image of $C$ under $f$ is a line or a circle.

(d) Let $C$ be the circle of radius $\sqrt{5}$ centered at $2 - 2i$. Find the image of $C$ under the map $f(z) = \frac{15}{z}$. In particular, find the center and radius of the new circle.

Note: In part (d), the center of $f(C)$ is not the point obtained by applying $f$ to the center of $C$, nor is there any automatic way of computing the new radius. Möbius transformations take circles to circles (or lines), but they don’t usually take the center of one circle to the center of another. Your best bet for (d) is to work directly with equations.