1. [2pts each]

Suppose that $p(z)$ is a polynomial with $\deg p \geq 2$, and let $f(z) = \frac{1}{p(z)}$.

(a) Explain why there is an $R_0 \in \mathbb{R}$ so that $f(z)$ is analytic (holomorphic) on the set $\{ z \mid |z| > R_0 \}$.

(b) Explain why, for every $R > R_0$, $\int_{|z|=R} f(z) \, dz = \int_{|z|=R_0} f(z) \, dz$.

(c) Estimate $|f(z)|$ on the circle $|z| = R$ (possibly just for large $R$) and use the estimate to show that $\int_{|z|=R_0} f(z) \, dz = 0$.

2. [2pts each]

(a) Give an example of a domain $D$ and an analytic (holomorphic) function $f(z)$ such that $|f(z)|$ achieves its minimum inside $D$, but $f(z)$ is not constant.

(b) Suppose that $D$ is a domain, and $f$ a function analytic (holomorphic) on $D$ which is never zero on $D$. Show that if $z_0 \in D$ is a local minimum for $|f(z)|$ then $f$ is constant near $z_0$.

3. [2pts] Prove a Liouville-type theorem for harmonic functions: If $u$ is function harmonic on all of $\mathbb{R}^2$, then if $u$ is bounded above or bounded below, $u$ must be constant.

**Hint:** Suppose that $u$ is a harmonic function defined on all of $\mathbb{R}^2$ and that $v$ is its harmonic conjugate. Let $f = u + iv$ be the corresponding entire function. What is $|\exp(f(z))|$?

4. [2pts] Show that the series $\sum_{k=1}^{\infty} \frac{\sin(kz)}{k^2}$ converges absolutely if $z \in \mathbb{R}$, but diverges if $z \in \mathbb{C} \setminus \mathbb{R}$.

5. [1pt each] Find the radius of convergence of the following power series:

(a) $\sum_{n=1}^{\infty} \frac{3^n}{n} z^n$

(b) $\sum_{n=0}^{\infty} \frac{2^n}{n!} z^{3n}$

(c) $\sum_{n=0}^{\infty} n! z^n$

(d) $\sum_{n=0}^{\infty} \frac{1}{n^2} (z + 1)^n$
6. [2pts each] Suppose that $f$ is an analytic function in the disk $D_R(z_0) = \{ z \mid |z - z_0| < R \}$ and has Taylor series expansion

$$f(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^k \text{ in } D_R(z_0)$$

(a) What is the formula for the coefficients $a_k$ in terms of the derivatives of $f$ at $z_0$?

(b) Suppose that $C_r$ is a circle centered at $z_0$ of radius $r < R$. Compute

$$\frac{k!}{2\pi i} \int_{C_r} \frac{f(z)}{(z - z_0)^{k+1}} \, dz$$

by using power series expansion for $f$ and exchanging integration and summation.