MATH/MTHE 326  SOLUTIONS TO MIDTERM EXAM B
NOVEMBER 1, 2016

INSTRUCTIONS: The exam has six questions labelled 1 through 6. Each question is worth 10 marks.
The exam is two hours in length.
To receive full credit you must explain your answers.
Write all answers on the exam. You may use the backs of pages if necessary.
Pre-approved calculator (Casio 991 or equivalent) is permitted.

Name: ________________________________

Student Number: ________________________________

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1. (a) [5pts] Let $z = -1 + i$ and $w = 1 + i\sqrt{3}$. Find the polar forms of $zw$ and $\frac{z}{w}$.

**Solutions:** We compute that $|z| = \sqrt{1 + 1} = \sqrt{2}$ and

$$z = -1 + i = \sqrt{2}(\cos(\theta) + i\sin(\theta))$$

where

$$\cos(\theta) = -\frac{1}{\sqrt{2}} \quad \text{and} \quad \sin(\theta) = \frac{1}{\sqrt{2}}.$$

So

$$\theta = \frac{3\pi}{4} + 2k\pi \quad \text{with} \quad k \in \mathbb{Z}.$$

The polar form of $z = -1 + i$ is

$$z = -1 + i = \sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right).$$

Similarly, we compute that $|w| = \sqrt{3 + 1} = 2$ and

$$w = 1 + i\sqrt{3} = 2(\cos(\tau) + i\sin(\tau))$$

where

$$\cos(\tau) = \frac{1}{2} \quad \text{and} \quad \sin(\tau) = \frac{\sqrt{3}}{2}.$$

So

$$\tau = \frac{\pi}{3} + 2\ell\pi \quad \text{with} \quad \ell \in \mathbb{Z}.$$

The polar form of $w = 1 + i\sqrt{3}$ is

$$w = 1 + i\sqrt{3} = 2 \left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right).$$

Now

$$zw = (-1 + i)(1 + i\sqrt{3}) = |z||w|(\cos(\theta + \tau) + i\sin(\theta + \tau))$$

$$= 2\sqrt{2} \left(\cos\left(\frac{3\pi}{4} + \frac{\pi}{3}\right) + i\sin\left(\frac{3\pi}{4} + \frac{\pi}{3}\right)\right)$$

$$= 2\sqrt{2}\left(\cos\left(\frac{13\pi}{4}\right) + i\sin\left(\frac{13\pi}{4}\right)\right).$$

and

$$\frac{z}{w} = \frac{|z|}{|w|}(\cos(\theta - \tau) + i\sin(\theta - \tau))$$
\[
\frac{\sqrt{2}}{2} \left( \cos\left(\frac{3\pi}{4} - \frac{\pi}{3}\right) + i \sin\left(\frac{3\pi}{4} - \frac{\pi}{3}\right) \right) = \frac{1}{\sqrt{2}} \left( \cos\left(\frac{5\pi}{12}\right) + i \sin\left(\frac{5\pi}{12}\right) \right). 
\]

(b) [5pts] Find the possible values of \((1 + i)^i\).

**Solution:**

We have

\[(1 + i)^i = e^{i \log(1+i)}.\]

First note that

\[\log(1 + i) = \ln(\sqrt{2}) + i \arg(1 + i) = \ln(\sqrt{2}) + i\left(\frac{\pi}{4} + 2k\pi\right) \quad \text{for } k \in \mathbb{Z}.\]

So

\[(1 + i)^i = e^{i\left(\ln(\sqrt{2}) - \left(\frac{\pi}{4} + 2k\pi\right)\right)} = e^{-\frac{\pi}{4} - 2k\pi} e^{i\ln(\sqrt{2})}.\]

There are infinitely many values for \((1 + i)^i\).
2. (a) [5pts] Let \( f(z) = \frac{z+1}{z-i} \). Find the domain of \( f \). Find the set of all solution of the equation \( |f(z)| = 2 \).

**Solutions:** The function \( f \) is not defined when the denominator becomes zero. Thus, the domain of \( f \) is \( \mathbb{C} \setminus \{z \in \mathbb{C} \mid z = i\} \).

Now \( |f(z)| = 2 \Leftrightarrow |f(z)|^2 = 4 \) so that
\[
|z + 1|^2 = 4|z - i|^2
\]
Writing \( z = x + iy \), this equation is read as
\[
(x+1)^2 + y^2 = 4[x^2 + (y-1)^2] \Leftrightarrow x^2 + y^2 + 2x + 1 = 4(x^2 + y^2 - 2y + 1) \Leftrightarrow 3x^2 + 3y^2 - 2x - 8y + 3 = 0.
\]
Hence the equation \( |f(z)| = 2 \) represents the circle
\[
(x - \frac{1}{3})^2 + (y - \frac{4}{3})^2 = \frac{8}{9}
\]
It is a circle centered at \( \left(\frac{1}{3}, \frac{4}{3}\right) \) of radius \( \frac{2\sqrt{2}}{3} \).

(b) [5pts] with \( f(z) \) as in (a), find the inverse \( f^{-1}(z) \) and its domain. What is the image of the set \( \{z \in \mathbb{C} \mid |z| \leq 1\} \) under \( f^{-1} \).

**Solutions:** Put \( w = \frac{z + 1}{z - i} \). Then \( w(z - i) = z + 1 \). Solve for \( z \) in terms of \( w \):
\[
(w - 1)z = 1 + wi \Rightarrow z = \frac{1 + wi}{w - 1}.
\]
Hence
\[
 f^{-1}(z) = \frac{1 + z i}{z - 1},
\]
and its domain is \( \mathbb{C} \setminus \{z \in \mathbb{C} \mid z = 1\} \). We note that \( f^{-1} \) is a Möbius transformation.

We see that \( f^{-1} \) maps \( z = 1 \) to \( \infty \) and \( z = -\frac{1}{i} = i \) to 0. Put \( S = \{z \in \mathbb{C} \mid |z| = 1\} \). Then \( S \) represents the unit circle centered at the origin. By a property of Möbius transformations, \( S \) is mapped onto some straight line. To find which straight line it is, put \( z = -1 \), which is in \( S \). Then \( f^{-1}(-1) = \frac{\frac{1}{2} + i}{\frac{1}{2} - i} = \frac{1 - i}{2} \), so \( S \) is mapped onto the line \( y = -x \). To see which half-plane is the image of the interior \( |z| < 1 \) of \( S \), \( z = 0 \) is obviously in the interior of \( S \). Then \( f^{-1}(0) = -1 \). This says that the interior of \( S \) is in the lower half of the plane under the line \( y = -x \). Therefore, the set \( \{z \in \mathbb{C} \mid |z| \leq 1\} \) is mapped to the lower half-complex plane under the line \( y = -x \).
3. Let \( z = x + iy \), and let \( f(z) = u(x, y) + iv(x, y) \) be a function of \( z \). Suppose that \( v(x, y) = e^{-2x} \sin(2y) \) is given.

(a) [3pts] Is \( v \) harmonic?

**Solution:** The first partials are

\[
v_x = -2e^{-2x} \sin(2y); \quad v_y = 2e^{-2x} \cos(2y).
\]

The second partials are:

\[
v_{xx} = 4e^{-2x} \sin(2y); \quad v_{yy} = -4e^{-2x} \sin(2y).
\]

Then the Laplacian of \( v \) is:

\[\nabla^2 v = v_{xx} + v_{yy} = 0.\]

Therefore, \( v \) is harmonic.

(b) [5pts] Find a harmonic conjugate \( u \) of \( v \).

**Solution:** By the Cauchy–Riemann equation, we have the identity: \( u_x = v_y \), so,

\[u_x = 2e^{-2x} \cos(2y)\]

Integrating with respect to \( x \), we get

\[
u = \int_0^x 2e^{-2t} \cos(2y) \, dt = -e^{-2x} \cos(2y) + g(y)
\]

where \( g(y) \) is a differentiable function of \( y \).

By the other Cauchy–Riemann equation, we have the identity: \( u_y = -v_x \), so,

\[
\begin{align*}
u_y &= 2e^{-2x} \sin(2y) + g'(y) = -v_x = 2e^{-2x} \sin(2y).
\end{align*}
\]

Hence \( g'(y) = 0 \). Thus \( g(y) = C \) is a constant. So

\[u(x, y) = -e^{-2x} \cos(2y) + C.\]

[2pts] (c) Show that \( u \) is also harmonic.

**Solution:** We have

\[
u_x = 2e^{-2x} \cos(2y); \quad u_{xx} = -4e^{-2x} \cos(2y)
\]

and

\[
u_y = 2e^{-2x} \sin(2y); \quad u_{yy} = 4e^{-2x} \cos(2y).
\]

Then the Laplacian of \( u \) is:

\[\nabla^2 u = u_{xx} + u_{yy} = 0.\]

Therefore, \( u \) is harmonic.
4. (a) [3pts] Let $f(z)$ be a complex-valued function defined in an open neighborhood $D$ about $z_0$. What is the necessary condition for $f$ to be analytic on $D$?

**Solution:** The necessary condition for the analyticity of $f(z)$ in $D$ is that it satisfies the Cauchy–Riemann equations: Let $z = x + iy$ and $f(z) = u + iv$. Then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

for every point $(x, y)$ in $D$.

(b) [3pts] Let $f(z) = \frac{1}{z^2 + 2iz + 1}$. Find the domain where $f$ is analytic.

**Solution:** We note that $f(z)$ is not well-defined if $z^2 + 2iz + 1 = 0$. Solving this equation with the quadratic formula, we obtain

$$z = -i \pm \sqrt{(-i)^2 - 1} = -i \pm i\sqrt{2} = i(-1 \pm \sqrt{2}).$$

So the domain of $f$ is

$$D = \mathbb{C} \setminus \{z = i(-1 \pm \sqrt{2})\}.$$

(c) [4pts] Let $z = x + iy$. Let $f(z) = e^y \cos(x) + ie^y \sin(x)$ analytic?

**Solution:** We test if the Cauchy–Riemann equations hold or not. Here $u(x, y) = e^y \cos(x)$ and $v(x, y) = e^y \sin(x)$.

$$\frac{\partial u}{\partial x} = -e^y \sin(x), \quad \frac{\partial u}{\partial y} = e^y \cos(x).$$

Similarly,

$$\frac{\partial v}{\partial x} = e^y \cos(x), \quad \frac{\partial v}{\partial y} = e^y \sin(x).$$

Hence the Cauchy–Riemann equations are satisfied when

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

These implies that

$$-e^y \sin(x) = e^y \sin(x) \quad \Rightarrow \quad \sin(x) = 0$$

and

$$e^y \cos(x) = -e^y \cos(x) \quad \Rightarrow \quad \cos(x) = 0.$$ 

These cannot be satisfied for the same values of $x$. Hence $f$ cannot be analytic anywhere.
5. (a) [5pts] Compute 
\[ \int_{\gamma} \frac{1}{z} \, dz, \]
where \( \gamma \) is given by the ellipse \( 4x^2 + y^2 = 1 \), traversed once clockwise.

**Solution:** The integrand \( \frac{1}{z} \) is analytic in the plane with origin removed. Also the ellipse \( 4x^2 + y^2 = 1 \) can be deformed continuously without passing through the origin to the circle \( \gamma_0 : x^2 + y^2 = 1 \) traversed once clockwise. Hence
\[ \int_{\gamma} \frac{1}{z} \, dz = \int_{\gamma_0} \frac{1}{z} \, dz = -2\pi i. \]

(b) [5pts] Find the residues of \( f(z) = \frac{4}{4z^2 + 17z + 4} \) at its poles.

**Solution:** We factor the polynomial \( 4z^2 + 17z + 4 \) into the product of linear polynomials.
\[ 4z^2 + 17z + 4 = 4(z^2 + \frac{17}{4})(z + 1) = 4(z + \frac{1}{4})(z + 4). \]
So \( f(z) \) has poles at \( z = -\frac{1}{4} \) and \( z = -4 \). Then
\[ f(z) = \frac{4}{4(z + \frac{1}{4})(z + 4)} = \frac{A}{z + \frac{1}{4}} + \frac{B}{z + 4} = \frac{A(z + 4) + B(z + \frac{1}{4})}{(z + \frac{1}{4})(z + 4)}. \]
We have the identity
\[ 1 = A(z + 4) + B(z + \frac{1}{4}). \]
Take \( z = -4 \) in this equation, we get
\[ 1 = B(-4 + \frac{1}{4}) = B(-\frac{15}{4}) \quad \Rightarrow \quad B = -\frac{4}{15}. \]
Similarly, take \( z = -\frac{1}{4} \) in this equation, we get
\[ 1 = A(-\frac{1}{4} + 4) = A(\frac{15}{4}) \quad \Rightarrow \quad A = \frac{4}{15}. \]
The residues are then
\[ \text{Res}(-\frac{1}{4}) = \frac{4}{15}, \text{Res}(-4) = -\frac{4}{15}. \]
6. (a) [5pts] Calculate
\[ \int_\gamma |z|^2 \, dz, \]
where \( \gamma \) is the line segment with the initial point \(-1\) and the terminal point \(i\).

**Solution:** Parametrize the line segment by
\[ z = z(t) = -1 + (1 + i)t, \quad 0 \leq t \leq 1 \]
so that
\[ |z|^2 = (-1 + t)^2 + t^2 \quad \text{and} \quad dz = (1 + i)dt. \]
The value of the integral becomes
\[
\int_\gamma |z|^2 \, dz = \int_0^1 (2t^2 - 2t + 1)(1 + i) \, dt \\
= (1 + i) \left[ \frac{2}{3} t^3 - t^2 + t \right]_0^1 = \frac{2}{3}(1 + i).
\]

(b) [5pts] Compute
\[ \int_\gamma \frac{ze^z}{z^2 - 4} \, dz \]
where \( \gamma \) is the circle of radius 3 centered at \(1 + i\) oriented counterclockwise.

**Solution:** **Step 1:** First find the partial fraction decomposition of \( \frac{z}{z^2 - 4} \).

\[
\frac{z}{z^2 - 4} = \frac{A}{z - 2} + \frac{B}{z + 2}.
\]
Then \( A = \frac{1}{2} \) and \( B = \frac{1}{2} \). Hence
\[
\frac{z}{z^2 - 4} = \frac{1}{2(z - 2)} + \frac{1}{2(z + 2)}.
\]

**Step 2:** Therefore,
\[
\int_\gamma \frac{ze^z}{z^2 - 4} \, dz = \int_\gamma \frac{e^z}{2(z - 2)} \, dz + \int_\gamma \frac{e^z}{2(z + 2)} \, dz.
\]
Now $z = 2$ is inside the circle but $z = -2$ is not. So by the Cauchy integral formula, the second integral is 0 and we only need to calculate the first integral. Take $f(z) = \frac{e^z}{z}$ and use the Cauchy integral formal to obtain
\[
\int_{\gamma} \frac{e^z}{2(z - 2)} \, dz = 2\pi i \frac{e^2}{2} = \pi ie^2.
\]
Therefore,
\[
\int_{\gamma} \frac{ ze^z}{z^2 - 4} \, dz = \pi ie^2.
\]
Alternatively, we have
\[
\int_{\gamma} \frac{ ze^z}{z^2 - 4} \, dz = \int_{\gamma} \frac{ ze^z}{z + 2} \, dz = 2\pi i \frac{2e^2}{2 + 2} = \pi ie^2.
\]