Sample Midterm I

There are four questions.

Be as neat as possible, clearly state your results

Student Number:
Problem 1 [40 Points]

Let $\mathbb{X}$ be a Hilbert space and $x, y \in \mathbb{X}$. Prove the following:

a) [15 Points]
\[ ||x + y||^2 + ||x - y||^2 = 2||x||^2 + 2||y||^2. \]

b) [15 Points]
\[ |\langle x, y \rangle| \leq (||x||)(||y||). \]

c) [10 Points]
\[ 2|\langle x, y \rangle| \leq ||x||^2 + ||y||^2. \]
Problem 2 [30 points]

Let $H$ be a finite dimensional Hilbert space and $\{v_1, v_2\}$ be two linearly independent vectors in $H$. Let $b_1, b_2 \in \mathbb{R}$. Show that, among all vectors $x \in H$, which satisfies

$$\langle x, v_1 \rangle = b_1,$$
$$\langle x, v_2 \rangle = b_2,$$
the vector $x^* \in H$ has the minimum norm if $x^*$ satisfies:

$$x^* = \alpha_1 v_1 + \alpha_2 v_2,$$

with

$$\langle v_1, v_1 \rangle \alpha_1 + \langle v_2, v_1 \rangle \alpha_2 = b_1,$$
$$\langle v_1, v_2 \rangle \alpha_1 + \langle v_2, v_2 \rangle \alpha_2 = b_2.$$

Hint: You may use any approach you wish to adopt. The Projection Theorem, however, is important.
Problem 3 [30 Points]

a) [15 Points] Let \( T \) be a mapping from \( L_2(\mathbb{R}^+; \mathbb{R}) \) to \( \mathbb{R} \) (extended to possibly include \(-\infty, \infty\)) given by:

\[
T(f) = \int_{\mathbb{R}^+} f(t) \frac{t}{1 + t^2} dt
\]

Let \( f_0 \in L_2(\mathbb{R}^+; \mathbb{R}) \) be given by:

\[
f_0(t) = \frac{1}{t^2 + 1}, \quad \forall t \in \mathbb{R}^+.
\]

Is \( T \) continuous on \( L_2(\mathbb{R}^+; \mathbb{R}) \) at \( f_0 \)?

b) [15 Points] Let \( S \) be the space of Schwartz signals. Let \( T : S \to \mathbb{R} \) be a mapping given by:

\[
T(\phi) = \phi'(0), \quad \phi \in S,
\]

where

\[
\phi'(t) = \frac{d}{dt}\phi(t) \quad \forall t.
\]

Is \( T \) a distribution on \( S \)? That is, is \( T \) continuous and linear on \( S \)?