Sample Midterm II

There are four questions.

Be as neat as possible, clearly state your results

Student Number:
Problem 1 [40 Points]

Let \( x, y \in \mathcal{f}(\mathbb{Z}; \mathbb{R}) \); that is, \( x, y \) map \( \mathbb{Z} \) to \( \mathbb{R} \), such that \( x = \{ \ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots \} \) and \( y = \{ \ldots, y_{-2}, y_{-1}, y_0, y_1, y_2, \ldots \} \) and \( x_k, y_k \in \mathbb{R} \), for all \( k \in \mathbb{Z} \).

For a)-c) below, state if the following are true or false with justifications in a few sentences:

a) \( \langle x, y \rangle = \sum_{i \in \mathbb{Z}} i^2 x_i y_i \) is an inner-product.

b) \( \langle x, y \rangle = \sum_{i \in \mathbb{Z}} x_i y_i \) is an inner-product.

c) \( \{ x : ||x||_2^2 < \infty \} \) contains a complete orthonormal sequence, where \( ||x||_2 = \sqrt{\sum_{i \in \mathbb{Z}} |x(i)|^2} \).

d) Consider a linear system described by the relation:

\[
y(n) = \sum_{m \in \mathbb{Z}} h(n, m) u(m), \quad n \in \mathbb{Z}
\]

for some \( h : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C} \). Show that such a system is time-invariant if and only if it is a convolution system, that is it can be written as:

\[
y(n) = \sum_{m \in \mathbb{Z}} h(n - m) u(m)
\]

for some \( h : \mathbb{Z} \to \mathbb{R} \).
Problem 2 [15 Points]

Does there exist a sequence of functions \( \{f_j\} \) in \( L_2(\mathbb{R}^+; \mathbb{R}) \) such that a sequence of distributions \( \tilde{f}_j \) represented by \( f_j \) on the set of Schwartz signals \( S \) converges to zero in a distributional sense, but \( f_j \) does not converge to zero (that is, in the \( L_2 \) norm). That is, does there exist a sequence of functions \( \{f_j\} \) in \( L_2(\mathbb{R}^+; \mathbb{R}) \) such that

\[
\lim_{j \to \infty} \left( \int_0^\infty |f_j(t)|^2 dt \right)
\]

is not zero, but

\[
\lim_{j \to \infty} \left( \int_0^\infty f_j(t)\phi(t)dt \right) = 0, \quad \forall \phi \in S.
\]

If there exists one, give an example. If there does not exist one, explain why.
Problem 3 [20 Points]

Alice and Bob are approached by a generous company and asked to solve the following problem: The company wishes to store any signal $f$ in $L_2(\mathbb{R}_+; \mathbb{R})$ in a computer with a given error of $\epsilon > 0$, that is for every $f \in L_2(\mathbb{R}_+)$, there exists some signal $h \in H$ such that $\|f - h\|_2 \leq \epsilon$ (thus the error is uniform over all possible signals), where $H$ is the stored family of signals (in the computer’s memory).

To achieve this, they encourage Alice or Bob to use a finite or a countable expansion to represent the signal and later store this signal in an arbitrarily large memory. Hence, they allow Alice or Bob to purchase as much memory as they would like for a given error value of $\epsilon$.

Alice turns down the offer and says it is impossible to do that for any $\epsilon$ with a finite memory and argues then she needs infinite memory, which is impossible.

Bob accepts the offer and says he may need a very large, but finite, memory for any given $\epsilon > 0$; thus, the task is possible.

Which one is the accurate assessment?

a) If you think Alice is right, which further conditions can she impose to make this possible? Why is she right?

b) If you think Bob is right, can you suggest a method? Why is he right?

Please be precise to receive credit. Even if your answer is wrong, a precise discussion will lead to partial credit.
Problem 4 [25 points]

Let \( x \) be in the real Hilbert space \( L_2([0, 1]; \mathbb{R}) \) with the inner product
\[
\langle x, y \rangle = \int_0^1 x(t)y(t)dt.
\]
We would like to express \( x \) in terms of the following two signals (which belong to the Haar signal space)
\[
\begin{align*}
u_1(t) &= 1_{\{t \in [0, 1/2]\}} - 1_{\{t \in [1/2, 1]\}}, \quad t \in [0, 1] \\
u_2(t) &= 1_{\{t \in [0, 1]\}}, \quad t \in [0, 1]
\end{align*}
\]
such that
\[
\int_0^1 \left| x(t) - \sum_{i=1}^2 \alpha_i u_i(t) \right|^2 dt
\]
is minimized, for \( \{\alpha_1, \alpha_2 \in \mathbb{R} \} \).

a) [10 Points] Using the Gram-Schmidt procedure, obtain two orthonormal vectors \( \{e_1(t), e_2(t)\} \) such that these vectors linearly span the same space spanned by \( \{u_1(t), u_2(t)\} \).

b) [5 Points] State the problem as a projection theorem problem by clearly identifying the Hilbert space and the projected subspace.

c) [10 Points] Let \( x(t) = 1_{\{t \in [1/2, 1]\}} \). Find the minimizing \( \alpha_1, \alpha_2 \) values.