Homework Assignment 2

Problem 1: Projection Theorem

Recall a problem that we solved in class on projection to a subspace. We now modify that problem with
the minimization of
\[
\int_{-1}^{1} t^2 (t^n - m(t))^2 dt
\]
over all \(m(t)\) such that
\[
m(t) \in M := \{ f : f \in L_2([-1, 1]; \mathbb{R}), f = \alpha + \beta t, \alpha, \beta \in \mathbb{R} \}
\]
a) State the problem as a Projection problem by identifying the Hilbert space, and the subspace.
b) Compute the solution, that is find the minimizing \(m(t)\).

Hint: Note that, the only difference here is that the inner product term has an additional \(t^2\) in it. In this
case the Hilbert space definition will differ only by the inner product the space entails.

Problem 2: A More General Projection Theorem [20 Points]

**Theorem:** Let \(M\) be a closed subspace of a Hilbert space \(H\). Let \(x\) be a fixed element in \(H\) and let \(V \subset H\) be a subset such that \(V = \{ v : x + y, y \in M \}\) (also called a linear variety of \(M\)). Then there is a unique
vector \(m_0 \in V\) of minimum norm. Furthermore, \(m_0\) is orthogonal to \(M\).

The proof of the above follows almost identically the discussion in class: Translate \(V\) by \(-x\) so that it
becomes a closed subspace and apply the Projection Theorem discussed in class.

The following is an application of this result: Let \(x \in \mathbb{R}^n\). Consider the following optimization problem:
\[
\min x^T Qx,
\]
such that
\[
Ax = b,
\]
where \(A\) is an \(m \times n\) matrix \(m \leq n\), with rank \(m\) and \(Q\) a symmetric, positive definite matrix.

Show that, using the Projection Theorem, the optimal \(x^*\) minimizing \(x^T Qx\) is given by:
\[
x^* = Q^{-1} A^T (A Q^{-1} A^T)^{-1} b
\]

Problem 3: Orthonormal Sequences

Show that the family of complex exponentials in \(L_2([0, 2\pi]; \mathbb{C})\):
\[
\{ e_n(t) \} = \left\{ \frac{1}{\sqrt{2\pi}} e^{int}, \quad n \in \mathbb{Z} \right\}
\]
forms an orthonormal sequence. This sequence is used for the Fourier expansion of functions in $L_2([0, 2\pi]; \mathbb{C})$.

Problem 4: Convergence in a Hilbert Space

Let $\{e_i\}$ be a sequence of orthonormal vectors in a Hilbert space $H$. Let $\{x_n = \sum_{i=1}^n \xi_i e_i\}$ be a sequence of vectors in $H$. Show that this sequence converges to a vector $x$ if and only if

$$\sum_{i=1}^\infty |\xi_i|^2 < \infty.$$

Problem 5: Separability

Show that $l_2(\mathbb{N}; \mathbb{R})$ is separable.

Problem 6: Orthonormal Sequences

Let $H$ be a Hilbert space, and $\{e_i\}$ a complete orthonormal sequence in $H$. That is, the only element in the Hilbert space which is orthogonal to each of the $e_i$ vectors is the null vector.

Show that there is a unique representation for every $h \in H$ in terms of linear expansions involving the sequence $\{e_i\}$.

Problem 7: Matlab Assignment (Gram-Schmidt Procedure)

In Matlab, generate a (function) code named GramSchmidtProcedure, which takes as input a fixed number of linearly independent vectors and generates a family of orthonormal vectors as a result. Apply your code to the following problem:

Let

$$x_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 0 \\ 4 \\ 2 \\ 1 \end{bmatrix}, \quad x_4 = \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

Generate a sequence of orthonormal vectors $\{e_1, e_2, e_3, e_4\}$ which span the same space that is spanned by $\{x_1, x_2, x_3, x_4\}$.

Problem 8: [Optional] Optimal Control through the Projection Theorem

Related to Problem 2 above, consider the following optimal control problem: Given a controlled system with dynamics:

$$x'(t) = u(t),$$

we wish to minimize the quadratic objective (cost)

$$J = \int_0^T (x^2(t) + u^2(t))dt,$$
where $x(0) \in \mathbb{R}$ is given. The goal is to find the best control policy $u(t)$ such that the cost function is minimized.

Express this as a Projection Theorem problem. You don’t need to solve the problem. Simply define the Hilbert space, inner-product and the projected subspace (or translated subspace / linear variety).

Hint: The constraint $x'(t) = u(t)$ can be written as $x(t) = x(0) + \int_0^t u(\tau)d\tau$. 