Homework Assignment 8

Problem 1 [Sampling Theorem, 20 Points]

a) Consider a discrete-time signal \( \{x(n)\} \) with a bandwidth \( B \). A discrete-time sampler samples this signal with a period \( N \) such that the sampled signal satisfies

\[
x_p(n) = \begin{cases} 
  x(n) & \text{if } 0 \equiv n \mod N, \\
  0 & \text{else.}
\end{cases}
\]

Following this, a decimator is applied to the system to obtain the signal:

\[x_d(n) = x_p(nN).\]

This new signal is stored in a storage device such as a recorder. Later, the original signal is attempted to be recovered from the storage device. What should the relation between \( B \) and \( N \) be such that, such a recovery is perfect, that is

\[
\sup_n |x(n) - \tilde{x}(n)| = 0,
\]

where \( \{\tilde{x}(n)\} \) denotes the reconstructed signal.

Identify the steps such that \( \{\tilde{x}(n)\} \) is recovered from \( \{x_d(n)\} \).

b) Typically human voice has a bandwidth of 4kHz. Suppose we wish to store a speech signal with bandwidth equal to 4kHz with a recorder. Since the recorder has finite memory, one needs to sample the signal. What is the maximum sampling period (in seconds) to be able to reconstruct this signal with no error.

Problem 2

Consider an impulse train defined by:

\[w_P(t) = \sum_{n \in \mathbb{Z}} \delta(t + nP)\]

so that the distribution that we can associate with this impulse train would be defined by:

\[
\overline{w_P}(\phi) = \sum_{n \in \mathbb{Z}} \phi(nP),
\]

for \( \phi \in \mathcal{S} \).

a) Show that \( \overline{w_P} \) is a distribution.

b) [Optional] Show that

\[
\hat{w}_P(\phi) = \int \frac{1}{P} w_P(t) \phi(t) dt,
\]
that is, the $F_{CC}$ of this train is another impulse train.

Hint: Take a look at the solutions to Assignment 6.

**Problem 3**

Consider a discrete-time signal $\{h(n)\}$ with DCFT as:

$$\hat{h}(f) = 1_{\{f \in [0,1/4] \cup (3/4,1)\}} f \in [0,1).$$

Determine the DCFT of the signal $g(n) = h(3n), n \in \mathbb{Z}$.

**Problem 4 [Applications in Communications]**

a) Let $m(t)$ be a real-valued signal with a bandwidth B. One can use a transformation, known as the Hilbert transform, to further compress the signal. This transform makes use of the fact that, the Fourier transform of a real signal is conjugate symmetric.

Let $\hat{x}$ denote the Hilbert transform of a signal $x$ in $L_2$, the space of square integrable functions with the usual inner product. The CCFT of the Hilbert transform of a signal is given by:

$$\hat{\hat{x}}(f) = -i \text{sign}(f) \hat{x}(f).$$

Using this relation, prove that the Hilbert transform of a signal is orthogonal to the signal itself.

b) Using your class notes, briefly describe the (double-sideband) Amplitude Modulation (AM) and the Frequency Modulation (FM) techniques for radio communications.

Using the result in part a), one can further suppress the bandwidth requirement for the double-sideband Amplitude Modulation (AM) technique, leading to a single-sideband AM signal.

* * * wish you a beautiful well-deserved summer break * * *