Problem 1: Distributional Derivatives
Show that the distributional derivative of the distribution represented by the unit step function is the delta distribution.

Solution:

The derivative of a distribution \(Df \langle g \rangle = -f \langle Dg \rangle\), where \(D(\cdot)\) denotes the derivative operator. Let \(u(t)\) denote the step function: that is \(u(t) = 1_{(t \geq 0)}\) (1(\_) being the indicator function). Hence,

\[
D\bar{u}(g) = -\bar{u}(Dg) = - \int_{\infty}^{0} (Dg)(t) = - \lim_{t \to \infty} g(t) + g(0) = g(0) = \bar{\delta}(g),
\]
for all \(g \in \mathcal{S}\) (In the above equation, \(g \in \mathcal{S}\) allows us to use that \(\lim_{t \to \infty} g(t) = 0\)).

Problem 2: Distributions
Let \(\mathcal{S}\) denote the space of Schwartz signals. Is the expression \(\int_{0}^{\infty} f(t) \phi(t) dt\) a distribution on \(\mathcal{S}\) for any given \(f \in L_1(\mathbb{R}_+; \mathbb{R})\)?

Solution:

We prove linearity and continuity. Note first that

\[
| \int_{0}^{\infty} f(t) \phi(t) dt | \leq \sup_{t} |f(t)| \int_{0}^{\infty} \phi(t) dt < \infty, \tag{1}
\]
for any \(\phi \in \mathcal{S}\). As a result, for any \(\alpha_1, \alpha_2 \in \mathbb{R}\) and \(\phi_1, \phi_2 \in \mathcal{S}\)

\[
\int_{0}^{\infty} f(t) (\alpha \phi_1(t) + \beta \phi_2(t)) dt = \alpha \int_{0}^{\infty} f(t) \phi_1(t) dt + \beta \int_{0}^{\infty} f(t) \phi_2(t) dt.
\]
Thus, \(\int_{0}^{\infty} f(t) \phi(t) dt\) defines a linear function on \(\mathcal{S}\).

Continuity follows since for \(\phi_n \to \phi \in \mathcal{S}\), we have that:

\[
| \int_{0}^{\infty} f(t) \phi_n(t) dt - \int_{0}^{\infty} f(t) \phi(t) dt | \leq \int_{0}^{\infty} | f(t) (\phi_n(t) - \phi(t)) | dt \leq \| f \|_1 \sup_{t \in \mathbb{R}} | \phi_n(t) - \phi(t) | \to 0.
\]

Problem 3: Approximate Identity Sequences [20 Points]
In class we observed that various approximate identity sequences exist and these can be used to define distributions that converge to \(\bar{\delta}\).

a) Recall that the sequence

\[
\psi_n(x) = c_n (1 + \cos(x))^n 1_{\{|x| \leq \pi\}}
\]
can be used to show that the complex harmonics of the form $e^{ikx}, k \in \mathbb{Z}$ forms a complete orthonormal sequence in $L_2([-\pi, \pi], \mathbb{C})$. Here, we take the normalizing coefficient $c_n$ to make $\int \psi_n(x)dx = 1$.

Show that
\[
\lim_{n \to \infty} \int_{|x| \geq \delta} \psi_n(x)dx = 0
\]
for all $\delta > 0$.

b) One useful sequence, which does not satisfy the non-negativity property that we discussed in class, but that satisfies the convergence property (to $\bar{\delta}$) is the following sequence:
\[
\psi_n(x) = \frac{\sin(nx)}{\pi x}
\]

Show that for any $\phi \in S$
\[
\lim_{n \to \infty} \int \psi_n(dx)\phi(x) = \phi(0)
\]

Hint: You can use the following results and hints:

- $\lim_{R \to \infty} \int_{-R}^{R} \frac{\sin(x)}{x} dx = \pi$
- Riemann-Lebesgue Lemma: For any integrable function $g$, $\lim_{|f| \to \infty} \int g(x) e^{ifx} dx = 0$.
- We can write $\phi(x) = \phi(0) + xh(x)$ for some smooth $h$.
- Express the integration as
\[
\int_{|x| \geq 1} \psi_n(x)\phi(x) + \int_{|x| \leq 1} \psi_n(dx)\phi(x)
\]
First, show that the first term goes to zero. For the second term, use the representation $\phi(x) = \phi(0) + xh(x)$ and the previous hints.

Solution:

a) Note that for $0 < A < B < \pi$,
\[
\int_{A}^{B} c_n(1 + \cos(x))^n \leq 1/2,
\]
and observe that
\[
c_n(1 + \cos(B))^n(B - A) \leq \int_{A}^{B} c_n(1 + \cos(x))^n \leq c_n(1 + \cos(A))^n(B - A)
\]

Thus,
\[
c_n \leq \frac{1}{2(B - A)} (1 + \cos(B))^{-n}
\]

Now, given $\delta > 0$, let $0 < A < B < \delta < \pi$. Then,
\[
\int_{|x| \geq \delta} \psi_n(x)dx \leq 2c_n(\pi - \delta)(1 + \cos(\delta))^n \leq \frac{1}{(B - A)} (1 + \cos(B))^{-n}(\pi - \delta)(1 + \cos(\delta))^n \to 0.
\]
b) This follows from the stated results by studying

\[ \int_{|x| \geq 1} \psi_n(x) \phi(x) + \int_{|x| \leq 1} \psi_n(dx) \phi(x) \]

The first expression goes to zero, by the Riemann-Lebesgue Lemma. For the second term, we write

\[ \int_{|x| \leq 1} \psi_n(x) \phi(x) = \int_{|x| \leq 1} \phi(0) \psi_n(x) dx + \int_{|x| \leq 1} \frac{\phi(x) - \phi(0)}{\pi x} \sin(nx) dx. \]

The second term in this expression goes to zero, by the Riemann-Lebesgue Lemma through the representation \( \phi(x) = \phi(0) + x h(x) \) for some smooth \( h \). The first term \( \int_{|x| \geq 1} \phi(0) \psi_n(x) dx \) converges to \( \phi(0) \): By writing \( u = n x \), we obtain

\[ \int_{|x| \leq 1} \frac{\sin(nx)}{\pi x} dx = \int_{|u| \leq n} \frac{\sin(u)}{\pi u} du \]

and using the fact that \( \lim_{n \to \infty} \int_{-n}^{n} \frac{\sin(x)}{\pi x} dx = 1 \), the result follows.

**Problem 4: Systems**

Consider a linear system described by the relation:

\[ y(n) = \sum_{m \in \mathbb{Z}} h(n, m) u(m), \quad n \in \mathbb{Z} \]

for some \( h : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C} \).

a) When is such a system causal?

b) Show that such a system is time-invariant if and only if it is a convolution system.

**Solution:**

a) A linear system is causal if at any given time \( n \in \mathbb{Z} \), future control actions cannot affect the current output of the system. As such \( h(n, m) = 0 \) for \( m > n \) for a causal (non-anticipative) system.

b) We first show that linearity and time-invariance implies being a convolution system. Suppose a linear system described by

\[ y(n) = \sum_{m} h(n, m) u(m), \]

is time-invariant. Let \( v(m) = u(m - \theta) \) for some \( \theta \in \mathbb{Z} \). Let the signal \( g \) be the output of the system when the input is the discrete-time signal \( v \). It follows that

\[ g(n) = \sum_{m \in \mathbb{Z}} h(n, m) v(m) \]
\[ = \sum_{m \in \mathbb{Z}} h(n, m) u(m - \theta) \]
\[ = \sum_{m' \in \mathbb{Z}} h(n, m' + \theta) u(m') \]

(2)

By time-invariance, it must be that \( g(n) = y(n - \theta) \) and \( g(n + \theta) = y(n) \). Thus,

\[ g(n + \theta) = \sum_{m' \in \mathbb{Z}} h(n + \theta, m' + \theta) u(m') = y(n) \]

(3)
Therefore, \( h(n + \theta, m + \theta) = h(n, m) \) for all \( n, m \) values, and for all \( \theta \) values. Therefore \( h(n, m) \) should only be a function of the difference \( n - m \).

For the other direction; let a convolution system be given by

\[
y(n) = \sum_{m \in \mathbb{Z}} h(n - m)u(m)
\]  

(4)

Let \( v(m) = u(m - \theta) \) for some \( \theta \in \mathbb{Z} \). Let the signal \( g \) be the output of the system when the input is the discrete-time signal \( v \). Then,

\[
g(n) = \sum_{m \in \mathbb{Z}} h(n - m)u(m - \theta)
\]

\[
= \sum_{m' \in \mathbb{Z}} h(n - m - \theta)u(m)
\]

\[
= \sum_{m' \in \mathbb{Z}} h(n - \theta - m)u(m)
\]

\[
= \sum_{m' \in \mathbb{Z}} h(n - \theta - m')u(m')
\]

\[
= y(n - \theta)
\]  

(5)

Thus, \( g(n) = y(n - \theta) \) for all \( n \) and \( \theta \) in \( \mathbb{Z} \). Thus, a convolution system is linear and time-invariant.

**Problem 5**

Let the following input-output systems be mappings from \( \Gamma(\mathbb{Z};\mathbb{R}) \) to \( \Gamma(\mathbb{Z};\mathbb{R}) \), with the following input output relations. State if the each of the following systems are linear, memoryless, time-invariant and non-anticipative.

a) \[ y(n) = n^2x(n), \quad n \in \mathbb{Z} \]

b) \[ y(n) = 5x(n - 1) + x(n - 2), \quad n \in \mathbb{Z} \]

c) \[ y(n) = 4x(n), \quad n \in \mathbb{Z} \]

d) \[ y(n) = x(n)x(n - 1) + x(n), \quad n \in \mathbb{Z} \]

**Solution:**

a) Linear, memoryless, time-varying, non-anticipative.

b) Linear, with memory, time-invariant, non-anticipative.

c) Linear, memoryless, time-invariant, non-anticipative.

d) Non-Linear, with memory, time-invariant, non-anticipative. Non-linear because, a scaling in the input leads to a non-linear scaling in the output since if the input is scaled by \( \alpha \), the output is not scaled by \( \alpha \).

**Problem 6**

Recall the quantization operation from Homework Assignment 1. We can define a quantization system as follows:
Let \( C = \{0, \frac{1}{10}, \frac{2}{10}, \ldots, \frac{100}{10}\} \) be a discrete-time set and \( x \in \Gamma(C; \mathbb{R}) \) (that is \( x : C \to \mathbb{R} \)), and let \( B = \{0, H, \ldots, (N-1)H\} \). Let a quantizer \( Q \) map every element in \( \Gamma(C; \mathbb{R}) \) to \( \Gamma(C; B) \) pointwise in time, such that for all \( t \in C \), and \( y \in \Gamma(C; B) \), \( y(t) = Q(x(t)) \forall t \in C \), where \( Q : \mathbb{R} \to B \) denotes the pointwise quantization operation:

\[
Q(x(t)) = \begin{cases} 
0, & \text{if } x(t) < 0 \\
H\lfloor \frac{x(t)}{H} \rfloor, & \text{if } 0 \leq x(t) < (N-1)H \\
(N-1)H, & \text{if } x(t) \geq (N-1)H
\end{cases}
\]

(6)

Regarding the quantization operation as an input-output system, identify the input set and the output set and the relationship subset of the product set of the input and the output sets.

a) Is the quantization system \( Q \) defined above a memoryless one?

b) Is the quantization system \( Q \) time-invariant?

c) Is the quantization system \( Q \) linear?

**Solution:**

a) It is memoryless.

b) It is time-invariant.

c) It is not linear, hence non-linear: If the input is scaled, say, by two, the output is not scaled by two for \( x(n) > (N-1)H \).

**Problem 7**

Let \( x(t) \in \mathbb{R}^N \) and \( t \geq 0 \) and real-valued. Recall that the solution to the following differential equation:

\[
x'(t) = Ax(t) + Bu(t)
\]

with the initial condition \( x(0) = x_0 \) is given by

\[
x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau, \quad t \geq 0
\]

Suppose \( x(0) = 0 \) and \( u(t) = 0 \) for \( t < 0 \). Express the solution as a convolution

\[
x(t) = (h \ast u)(t),
\]

and find \( h(t) \).

**Solution:**

We can express:

\[
x(t) = \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau, \quad t \geq 0
\]

with the expression \( v = t - \tau, \; dv = -d\tau \), and that \( u(t) = 0 \) for \( t < 0 \)

\[
x(t) = \int_{v=0}^\infty e^{A(t-v)}Bu(t-v)dv, \quad t \geq 0
\]
Thus, with the representation
\[ y(t) = \int_{\tau} h(t - \tau) u(\tau) d\tau = \int h(v) u(t - v) dv, \]
it follows that
\[ h(t) = e^{At} B_1 \{ t \geq 0 \}. \]

Note: Recall that, in class, with the assumption that the system is stable, we avoided that condition that \( u(t) = 0 \) for \( t < 0 \). In that case, we were able to write
\[ x(t) = e^{At-t_0} x(t_0) + \int_{\tau=t_0}^{t} e^{A(t-\tau)} B u(\tau) d\tau, \]
leading to \( h(t) = e^{At} B_1 \{ t \geq 0 \} \). We will revisit the issue of stability later in the course.

**Problem 8**

Let \( x(n) \in \mathbb{R}^N \) and \( n \in \mathbb{Z} \). Consider a linear system given by
\[ x(n+1) = Ax(n) + Bu(n), \quad n \geq 0 \]
with the initial condition \( x(0) = 0 \). Suppose \( x(0) = 0 \) and \( u(n) = 0 \) for \( n < 0 \). Express the solution \( x(n) \) as a convolution
\[ x(n) = (h \ast u)(n), \]
and find \( h(n) \).

**Solution:**

We have that
\[ x(n) = \sum_{k=0}^{n-1} A^{n-k-1} Bu(k) = \sum_{s=1}^{n} A^{s-1} Bu(n-s) = \sum_{s=1}^{\infty} A^{s-1} Bu(n-s), \]
where we use \( u(n) = 0 \) for \( n < 0 \). Thus,
\[ h(n) = A^{n-1} B_1 \{n \geq 1\}. \]

Note: Recall that, in class, with the assumption that the system is stable, we avoided that condition that \( u(n) = 0 \) for \( n < 0 \). In that case, we were able to write
\[ x(n) = A^{n-n_0} x(n_0) + \sum_{m=n_0}^{n-1} A^{n-m-1} Bu(m), \]
leading to \( h(n) = A^{n-1} B_1 \{n \geq 1\} \). We will revisit the issue of stability later in the course.

**Problem 9 [Some Control Theory]**

Consider the system in Problem 8. Suppose \((A, B)\) is a controllable pair (Recall from Math 332 that, a system is controllable if with an appropriate input sequence \( \{u(n)\} \), the state of the system can be moved from any point in \( \mathbb{R}^N \) to any other point in \( \mathbb{R}^N \) in finite time). Let us extend the definition of bounded input bounded output (BIBO) stability to be discussed in class to include multi-dimensional inputs and outputs as follows: A controllable system is BIBO stable if \( \|u\|_{\infty} := \sup_{m \in \mathbb{Z}} \|u(m)\|_{\infty} < \infty \) implies that \( \|x\|_{\infty} := \sup_{m \in \mathbb{Z}} \|x(m)\|_{\infty} < \infty \). Show that the system is BIBO stable if and only if
\[ \max_{\lambda_i} |\lambda_i| < 1, \]
where \( \{\lambda_i\} \) is the set of eigenvalues of \( A \). You may assume that the system starts at time 0, with an initial condition of 0.

**Solution:**

We first prove that if
\[
\max_{\lambda_i} |\lambda_i| < 1,
\]
then the system is BIBO stable. Let \( \max_{\lambda_i} |\lambda_i| = \lambda \). Observe that, the system at any time can be written as:
\[
y(n) = \sum_{k=0}^{n} A^k Bu(n-k)
\]
We have that, by the Jordan form transformation of a matrix, \( A = SJS^{-1} \), we can write the above relationship as
\[
y(n) = \sum_{k=0}^{n} A^{(n-k)} Bu(k) = \sum_{k=0}^{n} SJ^{(n-k)} S^{-1} Bu(k)
\]
Let
\[
v(k) = S^{-1} Bu(k).
\]
We write
\[
y(n) = \sum_{k=0}^{n} SJ^{(n-k)} v(k) = S \sum_{k=0}^{n} J^{(n-k)} v(k)
\]
If \( \|u(k)\|_{\infty} < K \) for some \( K \), then it follows that,
\[
\|v(k)\|_{\infty} < M,
\]
for some \( M \). Now,
\[
\left\| \sum_{k=0}^{n} J^{(n-k)} v(k) \right\|_{\infty} \leq \sum_{k=0}^{n} \|J^{(n-k)} v(k)\|_{\infty} \leq \sum_{k=0}^{n} \|J^{(n-k)} M 1\|_{\infty}
\]
where \( J^{(n-k)} \) denotes the componentwise absolute value of the matrix, and \( 1 \) is an \( 1 \times n \) matrix of ones. Observe the following: Any Jordan form block \( J \) of size \( N \times N \), with eigenvalue \( \lambda_i \), can be written as
\[
\lambda_i I + E
\]
where \( E \) is a matrix which has all terms zero, except the super-diagonal (the points right above the diagonal), at which points the value is 1. For example for a three by three matrix, with eigenvalue 2, we can write the system as
\[
\begin{bmatrix}
2 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 2 \\
\end{bmatrix} + \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
\end{bmatrix}
\]
The second term \( E \) is such that \( E^N = 0 \) (in the example here, \( N = 3 \)). Finally, we use the power expansion and using the fact that any matrix commutes with the identity matrix:
\[
(\lambda_i I + E)^n = \sum_{k=0}^{n} \binom{n}{k} \lambda_i^n E^{n-k}.
\]
Since $E^N = 0$, we have
\[(\lambda I + E)^n = \sum_{k=0}^{N-1} \binom{n}{k} \lambda_I^{n-k} E^k\]

On the other hand, $\binom{n}{k} \lambda_I^{n-k}$, for every $k \leq N - 1$, decays to zero as $n \to \infty$, geometrically fast. Thus, $\sum_{k=0}^{n} \| J_1^{n-k} M1 \|_\infty$ is bounded.

For the converse, suppose the system is such that there exists an eigenvalue with $\lambda \geq 1$. First we discuss $\lambda = 1$. Suppose the applied input is such that the mode corresponding to the eigenvalue is excited (which is possible by the controllability assumption - had we not have this assumption, it might have been impossible to excite a given specified mode). By applying some n-dimensional sequence
\[
\{ u(LN), u(LN) \ldots, u(LN + N - 1), \quad L \in \mathbb{Z} \}
\]

one can excite the mode with eigenvalue 1 such that the system would grow unbounded with growing $n$. If $|\lambda| > 1$, even applying the control sequence to excite the unstable mode only once, and then applying zero control will lead the system to become unbounded with $n \to \infty$. 