MATH 337

Markov Chains

Course instructor: Dr. Scott Greenhalgh
Email: scott.greenhalgh@queensu.ca
Office: Jeff 516

Course website: http://www.mast.queensu.ca/~math337/index.shtml
Periodicity

Period of state $i$ is the largest (integer) $d$ s.t.

$$p_{ii}^{(n)} = 0 \text{ whenever } n \text{ is not divisible by } d.$$ 

**In other words**

Period: A state $i$ has period $d$ if any return to state $i$ occurs in multiples of $d$ steps

$$d = \gcd \{ n \mid p_{ii}^{(n)} > 0 \}$$
Review Ex.

What states are: Absorbing? Transient?

$$P = \begin{bmatrix}
0.25 & 0.75 & 0 & 0 & 0 \\
0.5 & 0.5 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0.33 & 0.67 & 0 \\
1 & 0 & 0 & 0 & 0
\end{bmatrix}$$
Expected # of time periods

Define

$$B_n = \begin{cases} 1 & \text{if } X_n = i \\ 0 & \text{o.w.} \end{cases}$$

The # of time periods in state $i$ given $X_0 = i$

$$\sum_{n=1}^{\infty} B_n | X_0 = i$$
First-passage times

Rather than the probability at ‘n’ steps of being in state $j$ given current state $i$, and alternative is to consider # of steps req’d to be in state $j$ given $i$.

Denote $f_{ij}^{(n)}$ as the prob of first passage time from $i$ to $j$ in ‘n’ steps
Expected first passage time

\[ \mu_{ij} = \begin{cases} 
\sum_{n=1}^{\infty} n \cdot f_{ij}^{(n)} & \text{if } \sum_{n=1}^{\infty} f_{ij}^{(n)} = 1 \\
\sum_{n=1}^{\infty} f_{ij}^{(n)} & \text{if } \sum_{n=1}^{\infty} f_{ij}^{(n)} < 1 
\end{cases} \]
Long-run properties of Markov Chains

Suppose we have an irreducible ergodic Markov Chain

$$\Rightarrow \lim_{n \to \infty} p_{ij}^{(n)} \text{ exists (and is indep of } i) = \pi_j$$

$$\pi_j \text{ is the steady state probability}$$
Absorption states

Recall \( p_{kk} = 1 \Rightarrow \) absorbing state

(once a markov chain visits state k it stays there)

First passage prob from i to k \(\Leftrightarrow\) prob of absorption into k