MATH 338 - Midterm Solutions

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1. We consider the following boundary value problem for the heat equation with $c = 1$ and $L = 1$:

$$u_t = u_{xx}\,, \qquad 0 < x < 1\,, \quad t > 0$$
$$u_x(0,t) = -u(0,t)\,, \quad u_x(1,t) = -u(1,t) \qquad \text{for all } t > 0$$
$$u(x,0) = f(x)$$

a) Using separation of variables, obtain

$$T' - kT = 0$$
$$X'' - kX = 0\,, \quad X'(0) = -X(0)\,, \quad X'(1) = -X(1)$$

b) Argue convincingly that k cannot be zero.

When $k = 0$, the DE for X has solution $X(x) = c_1 x + c_2$. The first boundary condition requires that $X'(0) + X(0) = c_1 + c_2 = 0$, so we must have $c_2 = -c_1$, and so $X(x) = c_1(x - 1)$. The second boundary condition requires that $X'(1) + X(1) = c_1 = 0$, so in fact $X(x) = 0$. Thus if $k = 0$, $X(x) = 0$ which would imply $u(x,t) = X(x)T(t) = 0$, and we are interested only in non-trivial solutions. So k cannot be zero.

c) Show that if $k = \mu^2$ with $\mu > 0$, then we must have $\mu = 1$ and the corresponding solutions are

$$T_0(t) = e^t\,, \qquad X_0(x) = e^{-x}$$

The general solution is $X(x) = c_1 \cosh \mu x + c_2 \sinh \mu x$. The first boundary condition requires that $X'(0) + X(0) = c_2\mu + c_1 = 0$, so $c_1 = -c_2\mu$, and so $X(x) = -c_2\mu \cosh \mu x + c_2 \sinh \mu x$. The second boundary condition requires that $X'(1) + X(1) = -c_2\mu^2 \sinh \mu + c_2 \sinh \mu = c_2(1 - \mu^2)\sinh \mu = 0$, which can only occur if $\mu = 1$. Thus the solution is, up to an arbitrary constant,

$$X = X_0 = \cosh x - \sinh x = \mathrm{e}^{-x}$$

The differential equation for T becomes $T' - T = 0$ with solution (up to an arbitrary constant) $T_0 = \mathrm{e}^t$.

d) Show that if $k = -\mu^2$ with $\mu > 0$, then $\mu = \mu_n = n\pi$, $n = 1, 2, \ldots$, and the corresponding solutions are

$$T_n(t) = \mathrm{e}^{-(n\pi)^2 t}$$
$$X_n(x) = n\pi \cos n\pi x - \sin n\pi x\,, \quad n = 1, 2, \ldots$$

The general solution is $X(x) = c_1 \cos \mu x + c_2 \sin \mu x$. The first boundary condition requires that $X'(0) + X(0) = c_2\mu + c_1 = 0$, so $c_1 = -c_2\mu$, and so $X(x) = -c_2\mu \cos \mu x + c_2 \sin \mu x$. The second boundary condition requires that $X'(1)+X(1) = c_2\mu^2 \sin \mu + c_2 \sin \mu = c_2(\mu^2+1) \sin \mu 0$, which is satisfied only when $\mu = \mu_n = n\pi$. Then the solutions are, up to an arbitrary constant,

$$X(x) = X_n(x) = n\pi \cos n\pi x - \sin n\pi x\,, \quad n = 1, 2, \ldots$$

With these values for $k = -\mu_n^2$, the differential equation for T becomes

$$T' + (n\pi)^2 T = 0$$

which has general solution, up to an arbitrary constant,

$$T_n(t) = \mathrm{e}^{-(n\pi)^2 t}\,, \quad n = 1, 2, \ldots$$

e) The functions $X_0, X_1, X_2, \ldots$ are orthogonal on the interval $(0, 1)$. Knowing what we do about orthogonality of trig functions, it's believable that $X_1, X_2, \ldots$ are orthogonal. Your job is to establish the orthogonality of X_0 with X_n for all $n \geq 1$; that is, show that

$$\int_0^1 X_0(x) X_n(x)\,\mathrm{d}x = 0\,, \quad n = 1, 2, \ldots$$

We find that

$$\int \mathrm{e}^{-x} \cos n\pi x \mathrm{d}x = \frac{\mathrm{e}^{-x}(-\cos n\pi x + n\pi \sin n\pi x)}{(n\pi)^2 + 1}$$

$$\int \mathrm{e}^{-x} \sin n\pi x \mathrm{d}x = \frac{\mathrm{e}^{-x}(-\sin n\pi x - n\pi \cos n\pi x)}{(n\pi)^2 + 1}$$

Therefore

$$\begin{aligned}
\int_0^1 X_0(x) X_n(x)\,\mathrm{d}x &= \int_0^1 \mathrm{e}^{-x}(n\pi \cos n\pi x - \sin n\pi x)\,\mathrm{d}x \\
&= \frac{\mathrm{e}^{-x}(-n\pi \cos n\pi x + n^2\pi^2 \sin n\pi x + \sin n\pi x + n\pi \cos n\pi x)}{(n\pi)^2 + 1} \Big|_0^1 \\
&= \frac{\mathrm{e}^{-x}((n^2\pi^2 + 1) \sin n\pi x)}{(n\pi)^2 + 1} \Big|_0^1 \\
&= \frac{\mathrm{e}^{-1}((n^2\pi^2 + 1) \sin n\pi) - \mathrm{e}^{0}((n^2\pi^2 + 1) \sin 0)}{(n\pi)^2 + 1} \\
&= \frac{0 - 0}{(n\pi)^2 + 1} \\
&= 0
\end{aligned}$$

f) Conclude that

$$u(x,t) = c_0 e^t e^{-x} + \sum_{n=1}^{\infty} c_n T_n(t) X_n(x)$$

where

$$c_0 = \frac{2e^2}{e^2-1} \int_0^1 f(x) e^{-x}\,\mathrm{d}x$$

and

$$c_n = \frac{2}{1+n^2\pi^2} \int_0^1 f(x) X_n(x)\,\mathrm{d}x\,, \quad n = 1, 2, \ldots$$

We've found, in effect, the following solutions to the heat equation with the above boundary conditions:

$$u_0(x,t) = T_0(t)X_0(t) = \mathrm{e}^t \mathrm{e}^{-x}$$

$$u_n(x,t) = T_n(t)X_n(t) = \mathrm{e}^{-(n\pi)^2 t}(n\pi\cos n\pi x - \sin n\pi x)\,, \quad n = 1, 2, \ldots$$

By linearity of the heat equation and boundary conditions, we can form a more general solution by taking an infinite linear combination of these:

$$u(x,t) = c_0 \mathrm{e}^t \mathrm{e}^{-x} + \sum_{n=1}^{\infty} c_n T_n(t) X_n(x)$$

Now, imposing the initial condition $u(x,0) = f(x)$ requires

$$f(x) = c_0 \mathrm{e}^{-x} + \sum_{n=1}^{\infty} c_n X_n(x)$$

We can compute the coefficients $c_0, c_1, c_2, \ldots$ using orthogonality. For c_0, multiply both sides by X_0 and integrate:

$$\begin{aligned}
\int_0^1 f(x) X_0(x)\,\mathrm{d}x &= \int_0^1 \left(c_0 \mathrm{e}^{-2x} + \sum_{n=1}^{\infty} c_n X_0(x) X_n(x) \right) \mathrm{d}x \\
&= c_0 \int_0^1 \mathrm{e}^{-2x}\,\mathrm{d}x + \sum_{n=1}^{\infty} c_n \int_0^1 X_0(x) X_n(x)\,\mathrm{d}x \\
&= c_0 \int_0^1 \mathrm{e}^{-2x}\,\mathrm{d}x + \sum_{n=1}^{\infty} c_n \cdot 0 \\
&= c_0 \frac{-\mathrm{e}^{-2x}}{2} \Big|_0^1 \\
&= c_0 \frac{1-\mathrm{e}^{-2}}{2} \\
&= c_0 \frac{\mathrm{e}^2 - 1}{2\mathrm{e}^2}
\end{aligned}$$

and so

$$c_0 = \frac{2\mathrm{e}^2}{\mathrm{e}^2 - 1} \int_0^1 f(x)\mathrm{e}^{-x}\,\mathrm{d}x$$

For c_n, multiply both sides by X_m and integrate:

$$\begin{aligned}
\int_0^1 f(x)X_m(x)\,\mathrm{d}x &= \int_0^1 \left(c_0\mathrm{e}^{-x}X_m(x) + \sum_{n=1}^{\infty} c_n X_n(x)X_m(x) \right) \mathrm{d}x \\
&= c_0 \int_0^1 \mathrm{e}^{-x}X_m(x)\,\mathrm{d}x + \sum_{n=1}^{\infty} c_n \int_0^1 X_n(x)X_m(x)\,\mathrm{d}x \\
&= c_0 \int_0^1 X_0(x)X_m(x)\,\mathrm{d}x + c_m \int_0^1 (X_m(x))^2\,\mathrm{d}x \\
&= 0 + c_m \frac{m^2\pi^2 + 1}{2}
\end{aligned}$$

and so, replacing m with n,

$$c_n = \frac{2}{n^2\pi^2 + 1} \int_0^1 f(x)X_n(x)\,\mathrm{d}x$$

2. *The Fourier series for*

$$f(x) = \begin{cases} 0 & -\pi < x < 1 \\ 1 & 0 < x < \pi \end{cases}$$

is

$$\frac{1}{2} + \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{\sin(2k+1)x}{2k+1}$$

Using this, and without directly computing the Fourier coefficients using the integral formulas*, write the Fourier series for*

$$g(x) = \begin{cases} -1 & -\pi < x < 1 \\ 1 & 0 < x < \pi \end{cases}$$

You can do this in at least two easy ways. Here's the first way: observe that $g(x) = 2g(x) - 1$. Then the Fourier series for $g(x)$ is

$$\frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\sin(2k+1)x}{2k+1}$$

Here's the second way: observe that $g(x) = f(x) - f(-x)$ (this coincides with the first way, since $-f(-x) = f(x) - 1$, in fact). Then the Fourier series for $g(x)$ is

$$\begin{aligned} \frac{1}{2} + \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{\sin(2k+1)x}{2k+1} - \frac{1}{2} - \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{\sin(-(2k+1)x)}{2k+1} \\ = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\sin(2k+1)x}{2k+1} \end{aligned}$$

since $\sin(-x) = -\sin x$.

3. *Consider a string of unit length which oscillates in accordance with the wave equation with $c = 1$. If the initial shape is $f(x) = 0$ and the initial velocity is $g(x) = -10$, use d'Alembert's formula to write the solution $u(x,t)$. Describe G, an antiderivative of g^*, completely, with a graph.*

The odd extension of $g(x)$ is

$$g^*(x) = \begin{cases} 10 & -1 < x < 0 \\ -10 & 0 < x < 1 \end{cases}$$

Now we find an expression for the antiderivative $G(x)$ of $g^*(x)$; remember that $G(x)$ is 2-periodic, so it suffices to determine $G(x)$ for x on an interval of length 2.

Is there a particular interval of length 2 on which $g^*(x)$ has a very simple form? Well, it actually doesn't get any simpler than $g^*(x)$ on $-1 < x < 1$, so we need $G(x)$ for $-1 < x < 1$. First, for $-1 < x < 0$, we find

$$\begin{aligned} G(x) &= \int_{-1}^{x} g^*(s)\,\mathrm{d}s \\ &= \int_{-1}^{x} 10\,\mathrm{d}s \\ &= 10s\,\big|_{-1}^{x} \\ &= 10x + 10 \end{aligned}$$

Next for $0 < x < 1$, we find

$$\begin{aligned} G(x) &= \int_{-1}^{0} g^*(s)\,\mathrm{d}s + \int_{0}^{x} g^*(s)\,\mathrm{d}s \\ &= 10 - \int_{0}^{x} 10\,\mathrm{d}s \\ &= 10 - 10x \end{aligned}$$

Putting it together,

$$G(x) = \begin{cases} 10 + 10x & -1 < x < 0 \\ 10 - 10x & 0 < x < 1 \\ G(x+2) & \text{otherwise} \end{cases}$$

With this G, and $c = 1$ and $f(x) = 0$, d'Alembert's solution is

$$u(x,t) = \frac{1}{2}(G(x+t) - G(x-t))$$