

Name: _____

MATH 338 - Quiz 7

1. Let $f(x) = \begin{cases} e^{-\frac{1}{x}} & 0 < x < \infty \\ 0 & -\infty < x \leq 0 \end{cases}$.

(a) Convince yourself that for any $x > 0$, the k th derivative $f^{(k)}(x)$ is a linear combination of terms of the form $\frac{e^{-\frac{1}{x}}}{x^m}$, m an integer.

This is straightforward.

(b) Show that $\frac{1}{x^{m+1}} \leq (m+1)! \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{x^n}$ and note that the right hand side is $(m+1)!e^{\frac{1}{x}}$.

Expanding the right hand side:

$$(m+1)! \left(\cdots + \frac{1}{(m+1)!} \frac{1}{x^{m+1}} + \cdots \right) = \cdots + \frac{1}{x^{m+1}} + \cdots$$

where the dots denote *positive* terms. So the inequality becomes clear.

(c) Conclude that $\frac{1}{x^m} \leq (m+1)!xe^{\frac{1}{x}}$; hence $\frac{e^{-\frac{1}{x}}}{x^m} \leq (m+1)!x$.

Multiply both sides of b) by x .

(d) Using (c), show that $\lim_{x \rightarrow 0^+} \frac{e^{-\frac{1}{x}}}{x^m} = 0$.

$$\lim_{x \rightarrow 0^+} \frac{e^{-\frac{1}{x}}}{x^m} \leq \lim_{x \rightarrow 0^+} (m+1)!x = 0$$

(e) Conclude that $f(x) \neq \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$

From the above, we conclude that $\lim_{x \rightarrow 0^+} f^{(k)}(x) = 0$. Furthermore, since $f(x) = 0$ for $x \leq 0$, it's clear that $\lim_{x \rightarrow 0^-} f^{(k)}(x) = 0$, and so $f^{(k)}(0) = 0$. Thus $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = 0$, and so this is not equal to the non-zero function $f(x)$.