Math 339 Winter 2017 Assignment 2 Solutions

Chapter 3 Question 3 (14 marks)

a) Strategy $t_3$ strictly dominates $t_1$ (1 mark), and so the first column of the matrix is deleted, leaving

\[
\begin{array}{c|cc}
&t_2&t_3 \\
\hline
s_1&(57,42)&(66,32) \\
s_2&(35,12)&(32,54) \\
s_3&(63,31)&(54,29) \\
\end{array}
\]

In this reduced game, $s_3$ dominates $s_2$ (1 mark), leaving

\[
\begin{array}{c|cc}
&t_2&t_3 \\
\hline
s_1&(57,42)&(66,32) \\
s_3&(63,31)&(54,29) \\
\end{array}
\]

Then $t_2$ dominates $t_3$ (1 mark) and then $s_3$ dominates $s_1$ (1 mark), leaving the strict NE at $(63, 31)$ or at $(s_3, t_2)$ (2 marks = 1 for the solution, and 1 for specifying it is strict).

b) $s_3$ dominates $s_4$, and $s_5$ (2 marks). $t_5$ dominates $t_1$, and $t_2$ (2 marks). Then $s_2$ dominates $s_1, s_3$ (2 marks), leaving the strictly dominant solution $(2,5)$, or written as $(s_2, t_3)$ (2 marks = 1 for the solution, and 1 for specifying it is strict).

Chapter 3 Question 4 (6 marks)

Consider the two by two payoff matrix in the general form:

\[
\begin{array}{c|cc}
(a,b)&(c,d) \\
(e,f)&(g,h) \\
\end{array}
\]

The conditions that force a two by two game to have no unique pure strategy NE solution are the following:

\[
a \geq e \text{ and } g \geq c \text{ and } b \geq d \text{ and } h \geq f
\]
The game reduces down to
\[
\begin{pmatrix}
(5,3) & (6,4) \\
(9,5) & (1,2)
\end{pmatrix}
\]
(2 marks for a correct reduction). The game is an irreducible two by two, and a MSNE exists. Begin by finding the payoffs to player 2 against player 1’s mixed strategy, if player 1 has probability \( p \) of using the first move (A) and \( 1 - p \) of using the second move (B).

\[
\pi_2(\sigma_1, C) = 3p + 5(1 - p), \quad \pi_2(\sigma_1, D) = 4p + 2(1 - p)
\]

2 marks for setting up the equations correctly, 3 marks for setting them equal to each other and solving for \( p \)

\[
5 - 2p = 2 + 2p \\
3 = 4p \rightarrow p = \frac{3}{4}
\]

The mixed strategy of player 1 is \( \sigma_1 = \frac{3}{4}(A) + \frac{1}{4}(B) \) (1 mark). To find the mixed strategy of player 2, we use a similar process. Let \( q \) be the probability that player 2 uses move their first move C, and \( 1 - q \) be the probability they use their second move D.

\[
\pi_c(A, \sigma_2) = 5q + 6(1 - q), \quad \pi_c(B, \sigma_2) = 9q + 1(1 - q)
\]
2 marks for setting up the equations correctly, 3 marks for setting them equal to each other and solving for \( q \)

\[ 6 - q = 1 + 8q \]

\[ 5 = 9q \rightarrow q = \frac{5}{9} \]

The mixed strategy of player 2 is \( \sigma_2 = \frac{5}{9}(C) + \frac{4}{9}(D) \) (1 mark). The expected payoff of both players at the MSNE is calculated as

\[
\pi_1(\sigma_1, \sigma_2) = \left( \frac{3}{4} \right) \left( \frac{5}{9} \right)(5) + \left( \frac{3}{4} \right) \left( \frac{4}{9} \right)(6) + \left( \frac{1}{4} \right) \left( \frac{5}{9} \right)(9) + \left( \frac{1}{4} \right) \left( \frac{4}{9} \right)(1) = 5.4
\]

\[
\pi_2(\sigma_1, \sigma_2) = \left( \frac{3}{4} \right) \left( \frac{5}{9} \right)(3) + \left( \frac{3}{4} \right) \left( \frac{4}{9} \right)(4) + \left( \frac{1}{4} \right) \left( \frac{5}{9} \right)(5) + \left( \frac{1}{4} \right) \left( \frac{4}{9} \right)(2) = 3 \frac{1}{3}
\]

(4 marks = 2 for each correct calculation).

Chapter 4 Question 2 (3 marks)

The game value is 3.

Chapter 4 Question 2 (25 marks)

The highest scoring strategy of standard Goofspiel when you know your opponent is going to play the strategy: \textit{match the value of the upturned card} is when they play K, you play A, then afterwards, lay down the card that is exactly one value higher than the card in the middle. The generalized version of this strategy is: imagine we relabel the playing cards as numerical versions of their worth, so A = 1, 2 = 2, ..., N = N, where the Nth card is the highest value in a finite deck. When the middle (spade) card is N, play your card valued at 1 (the A). For the rest of the game, play a card that is exactly one value higher than the card in the middle. (5 marks for stating this answer = 2 for the standard Goofspiel version + 3 for the generalized Goofspiel version). However, the bulk of the marks for this answer come
from the proof. There is probably more than one way to do it, but here is my proof.

Without loss of generality, imagine that the ordering of the play of a round of standard Goofspiel takes on the following form:

<table>
<thead>
<tr>
<th>You</th>
<th>Middle</th>
<th>Opponent</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>K</td>
<td>Q</td>
<td>Q</td>
</tr>
<tr>
<td>A</td>
<td>K</td>
<td>K</td>
</tr>
</tbody>
</table>

This gives a total of 65 points to you, and -65 points to the other player. It should not be hard to see that using the same strategy as your opponent, matching the value of the cards, will yield a score of zero for both players. The proof that the strategy is the highest scoring for both the standard game and the generalized game is completed by answering the following question, "Does there exist some order of play that will produce a score higher than 65 points in the standard game?"

The K can never be won, at best it can be matched, since your opponent will always put the K down when it shows up in the middle. If we decide to match the K, we are left with A through Q to use during the rest of the game. The highest score you can achieve in this situation is when you use your A against your opponent’s Q, and then lay down a card that is one value higher than the card in the middle for the rest of the plays in that round. Matching the K and losing the Q yields a payoff of 54 points to you, and -54 points to your opponent, which is clearly not better than 65.

Let $p$ be the value position of the card against which you decide to play your A. 1st value position is when you play your A against your opponent’s A, and 13th value position (in the standard game) is when you play your A against your opponent’s K. It should not be hard to see that once you have sacrificed your A to your opponent’s card, you will be able to win every card left in the middle deck that is smaller than the card you lost your A to, assuming you matched all the other high cards. If you lose the card at the $p$ position with your A, you can win positions $p - 1, p - 2, p - 3, ..., 2, 1 = \frac{(p-1)p}{2}$. Since Goofspiel is a zero sum game, your total score is $s(p) = \frac{(p-1)p}{2} - p = \frac{p^2}{2} - \frac{3}{2}p$. A quick calculation can show that $s(1) = 0, s(2) = -1, s(3) = 0$, and $s(4) = 2$, which are all less than optimal scores relative to 65 points in the standard game. An analysis of the score function $s(p)$, shows that it is increasing,
using the discrete form of the derivative:

\[ \Delta s = s(p + 1) - s(p) \]
\[ \Delta s = p - 1 \]

Which means that our score function is increasing when \( p > 1 \). (Note: If they use the derivative of a continuous function for a discrete function, take five marks off).

So if the size of the Goofspiel game is 1, 2, or 3 cards, it is best to use the matching strategy. However, when the size of the deck is greater than 3 cards, \( s(p) \) is maximized when the largest value of \( p \) is used to calculate it (and there is a largest value, since the deck size is finite), because \( s(p) \) is increasing. This means you should play your A when the highest value card in the deck is turned over, and then play cards exactly one value higher against every other card in the deck. Since \( s(p) \) can be calculated for any sized deck, this strategy is optimal for both the standard deck, and the finite one.

There might be other proofs that work, but the proof itself is worth 20 marks.