Assignment 3 Solutions

1. (a) We can write a general form for the payoffs in a 2x2 game quite succinctly. Observe: if player 1 plays strategy $A$ with probability $p$ ($B$ with prob. $(1-p)$) and player 2 plays $A$ with prob $q$ ($B$ with prob $(1-q)$), then for

$$
\begin{array}{c|cc}
 & A & B \\
\hline
A & a_1, a_2 & b_1, b_2 \\
B & c_1, c_2 & d_1, d_2
\end{array}
$$

the payoffs are given by

$$
\pi_1(p, q) = (a_1 - b_1 - c_1 + d_1)pq + (b_1 - d_1)p + (c_1 - d_1)q + d_1
$$

and

$$
\pi_2(q, p) = (a_2 - b_2 - c_2 + d_2)pq + (b_2 - d_2)p + (c_2 - d_2)q + d_2.
$$

We will use this general form for question two. It makes it easy, in a sense, since now we merely substitute in the appropriate terms. For 1a,

$$
\pi_1(p, q) = \frac{-7}{3}pq + \frac{5}{2}p - \frac{1}{2}q + \frac{1}{2}
$$

and

$$
\pi_2(q, p) = \frac{-3}{2}pq + \frac{3}{2}p + \frac{1}{2}q + \frac{1}{2}.
$$

(b) We have:

$$
\pi_1((p_1^*, p_2^*), (q_1^*, q_2^*)) = -6p_1q_1 - 8p_1q_2 + p_2q_1 - 7p_2q_2 + 5p_1 + p_2 + q_1 + 3q_2 - 1
$$

and

$$
\pi_2((q_1^*, q_2^*), (p_1^*, p_2^*)) = -6p_1q_1 - 7p_1q_2 - 5p_2q_1 - 10p_2q_2 + 5p_1 + 4p_2 + 2q_1 + 3q_2 - 1
$$

2. These are (at least four are) games from the first assignment. They may have an occuring role in our class. Recall: to solve these types of problems, we first eliminate any dominated strategies, and then revert to our file sharing example.

(a) There are no dominated strategies in this case. The expected payoffs are

$$
\pi_1(p, q) = 2pq - 1p + 2
$$

and

$$
\pi_2(q, p) = 2pq - 1q + 2.
$$

We now equate the following equations and solve for $q^*$:

$$
\pi_1(0, q^*) = 2
$$

$$
\pi_1(1, q^*) = 2q^* + 1.
$$

We find $q^* = 1/2$. Similarly,

$$
\pi_2(0, p^*) = 2
$$

$$
\pi_2(1, p^*) = 2p^* + 1.
$$

This yields $p^* = 1/2$. Hence, $(p^*, q^*) = (1/2, 1/2)$ is a MSNE. It is worth noting that the pure strategy Nash equilibria, $(A, A)$ and $(B, B)$ are, by the definition of mixed strategy Nash equilibria, still MSNE in the repeated game. However, for either of these two to be attained, coordination is required of the players. It is fine if the assignment you are grading lists only one MSNE, but it must be the non-pure strategy MSNE, if one exists. It is not OK if the student claim that no MSNE exists, since this is not at all correct.
(b) Start with the observation that, no matter what strategy player 1 is playing, player 2 will always receive a higher payoff playing strategy $B$ than it would playing strategy $A$. This means player 2 will never play strategy $A$. This is entirely equivalent to finding that $q = 0$; that is, the probability that player 2 plays $A$ is 0. Once we cross out strategy $A$ for player 2, we can use the same argument to eliminate strategy $A$ for player 1. This leaves us with only one strategy for each player. Hence, the MSNE is $(B, B)$. This is written in terms of probabilities as $(p^*, q^*) = (0, 0)$.

(c) Strategy $B$ is strictly dominated by strategy $A$. Hence, $(A, A)$ is the only MSNE. This is written in terms of probabilities as $(p^*, q^*) = (1, 1)$.

(d) There are no dominated strategies. Hence, we dive into the expected payoff calculations:

\[
\pi_1(p, q) = -\frac{5}{2}pq + 2p + q
\]

and

\[
\pi_2(q, p) = -\frac{5}{2}pq + 2q + p.
\]

We now equate the following equations and solve for $q^*$:

\[
\pi_1(0, q^*) = q^*
\]

\[
\pi_1(1, q^*) = -\frac{5}{2}q^* + 2 + q^*.
\]

We find $q^* = 4/5$. Similarly,

\[
\pi_2(0, p^*) = p^*
\]

\[
\pi_2(1, p^*) = -\frac{5}{2}p^* + p^* + 2.
\]

This yields $p^* = 4/5$. In all, $(p^*, q^*) = (4/5, 4/5)$ is an MSNE.

(e) Again, there are no dominated strategies.

\[
\pi_1(p, q) = -6pq + 3p + 6q - 4
\]

and

\[
\pi_2(q, p) = -4pq + 4p + 4q - 4.
\]

We now equate the following equations and solve for $q^*$:

\[
\pi_1(0, q^*) = 6q^* - 4
\]

\[
\pi_1(1, q^*) = -6q^* + 3 + 6q^* - 4.
\]

We find $q^* = 1/2$. Similarly,

\[
\pi_2(0, p^*) = 4p^* - 4
\]

\[
\pi_2(1, p^*) = -4p^* + 4p^* + 4 - 4.
\]

This gives $p^* = 1$. Hence, $(p^*, q^*) = (1, 1/2)$ is an MSNE.

3. We first look for dominated strategies. As we found in class on Wednesday, 25 January, $S_1$ is strictly dominated by $S_3$ for player 2. Hence, we can cross out $S_1$ for player 2. If we examine the resulting $3 \times 2$ game, we find that $S_2$ is a strictly dominated strategy for player 1. We get rid of that and are left with a $2 \times 2$ game,

<table>
<thead>
<tr>
<th></th>
<th>$S_2$</th>
<th>$S_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_1$</td>
<td>-2, 2</td>
<td>0, 0</td>
</tr>
<tr>
<td>$S_3$</td>
<td>2, -2</td>
<td>-1, 1</td>
</tr>
</tbody>
</table>
Now we can find the MSNE via the expected payoff functions,

\[ \pi_1(p, q) = -5pq + p + 3q - 1 \]

and

\[ \pi_2(q, p) = 5pq - p - 3q + 1. \]

where, in this case, player 1 plays strategy \( S_1 \) with probability \( p \) and \( S_3 \) with probability \((1 - p)\), and player 2 plays strategy \( S_2 \) with probability \( q \) and \( S_3 \) with prob. \((1 - q)\). We now equate the following equations and solve for \( q^* \):

\[ \pi_1(0, q^*) = 3q^* - 1 \]
\[ \pi_1(1, q^*) = 5q^* + 1 + 3q^* - 1. \]

We find \( q^* = 1/5 \). Similarly,

\[ \pi_2(0, p^*) = -p^* + 1 \]
\[ \pi_2(1, p^*) = 5p^* - p^* - 3 + 1. \]

Which yields \( p^* = 3/5 \). Hence, \((p^*, q^*) = (3/5, 1/5)\) is an MSNE. We could also write this MSNE as \(((3/5, 0, 2/5), (0, 1/5, 4/5))\).

4. I’ll grade this one. I encourage you to read the responses through, though, if you come across any.