Assignment 6 Solutions

The main point of this assignment is for you to develop an appreciation for equilibria. What exactly is an equilibrium? What does it mean for one to be stable? Unstable? You will also note that question one is really straight-forward; there is essentially no work to do. Question two is far more involved. This is the true nature of scientific/mathematical work: any deviation, no matter how slight, from the basic cases requires a good deal of work to analyze.

1. For this question, I list the equilibria and justify their stability. I have plotted the solutions in a separate file. **This question is worth 8/20.** Three marks should be given for finding the correct equilibria (all of them). The other five should be given for an adequate justification for the stability of these equilibria. This can take two forms (perhaps more): a) the derivative of the right-hand-side of the differential equation can be evaluated at the equilibria, as per Carly’s notes, or, b) a phase portrait can be plotted. I do both here, but only one is required.

(a) \[ \frac{dx}{dt} = 0 \Rightarrow \cos(2x) = \sin(2x) \]
Hence, the equilibria are \( x^* = \pi/8, 5\pi/8 \). Check the stability of these equilibria by evaluating \((\cos(2x) - \sin(2x))' = -2(\sin(2x) + \cos(2x))\) at those points. For \( x^* = \pi/8 \),
\[ -2(\sin(\pi/4) + \cos(\pi/4)) = -2\sqrt{2} < 0. \]
Hence, \( x^* = \pi/8 \) is stable. For \( x^* = 5\pi/8 \),
\[ -2(\sin(5\pi/4) + \cos(5\pi/4)) = 2\sqrt{2} > 0. \]
Hence, \( x^* = 5\pi/8 \) is unstable.

(b) \[ \frac{dx}{dt} = 0 \Rightarrow e^x = 2 \Rightarrow x^* = \ln(2) \]
Check for stability by using \((e^x - 2)' = e^x\).
\[ e^{\ln(2)} = 2 > 0. \]
Hence, \( x^* = \ln(2) \) is unstable.

(c) \[ \frac{dx}{dt} = 0 \Rightarrow x^3 - 3x + 2 = 0 \Rightarrow x^* = -2, 1. \]
Check for stability by using \((x^3 - 3x + 2)' = 3x^2 - 3\). For \( x^* = -2 \),
\[ 3(-2)^2 - 3 = 9 > 0. \]
Hence, \( x^* = -2 \) is unstable. For \( x^* = 1 \),
\[ 3(1)^2 - 3 = 0. \]
In this case, the derivative test fails. If you plot this question, you’ll see that \( x^* = 1 \) is unstable.
(d) \[
\frac{dx}{dt} = 0 \Rightarrow x \ln(x) = -1/4 \Rightarrow x \cong 0.6995, 0.1161
\]

Check for stability by using \((x \ln(x) + 1/4)' = \ln(x) + 1\). For \(x^* = 0.6995\),

\[
\ln(0.6995) + 1 > 0.
\]

Hence, \(x^* = 0.6995\) is unstable. For \(x^* = 0.1161\),

\[
\ln(0.1161) + 1 < 0.
\]

Hence, \(x^* = 0.1161\) is stable.

(e) \[
\frac{dx}{dt} = 0 \Rightarrow x + \cos(x) = 1 \Rightarrow x^* = 0
\]

Check for stability by using \((x + \cos(x) - 1)' = -\sin(x)\),

\[
1 - \sin(0) = 1 > 0.
\]

Hence, \(x^* = 0\) is unstable.

2. To answer these questions, we need to find the \(x\) and \(y\) nullclines, find where they intersect\(^1\), and evaluate the Jacobian at each of these points. **This question is worth 12/20.** Give six marks for finding all the equilibria and six marks for finding the stability of each of these.

(a) This is the most difficult of all the five. The reason is because one of the \(y\) nullclines is an ellipse. Let’s see how that works out:

x nullclines:

\[
x' = 0 \Rightarrow x(1-x+y) = 0 \Rightarrow x = 0, x-y = 1.
\]

y nullclines:

\[
y' = 0 \Rightarrow y(2-y^2-3x^2) = 0 \Rightarrow y = 0, (2-y^2-3x^2) = 0.
\]

That second \(y\) nullcline is the ellipse I told you about. We now need to find the intersection of the \(x\) and \(y\) nullclines. I’ll denote the nullclines:

\[N_{x1} := x = 0, \quad N_{x2} := x-y = 1, \quad N_{y1} := y = 0, \quad N_{y2} := 2-y^2-3x^2 = 0.\]

The intersection of the nullclines are, \((\cap)\) denotes intersection:

\[N_{x1} \cap N_{y1} = (0,0), \quad N_{x2} \cap N_{y1} = (1,0)\]

\[N_{x1} \cap N_{y2} = (0, \pm \sqrt{2}), \quad N_{x2} \cap N_{y2} = \left(\frac{1+\sqrt{5}}{4}, \frac{-3+\sqrt{5}}{4}\right)\]

I found the last two equilibria by substituting the equation for \(N_{x2}\) into the equation for \(N_{y2}\) and solving the resulting quadratic.

Now for stability. The Jacobian is

\[
J(x,y) = \begin{bmatrix}
-2x + 1 + y & x \\
-6xy & 2 - 3y^2 - 3x^2
\end{bmatrix}
\]

\(^1\)Where the \(x\) and \(y\) nullclines intersect. Not where the \(x\) (or \(y\)) nullclines intersect themselves.
The following useful chart summarizes the stability of the six (yowza!) equilibria:

<table>
<thead>
<tr>
<th></th>
<th>Det</th>
<th>Tr</th>
<th>((Tr^2 - 4Det))</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(0,0)</td>
<td>2</td>
<td>3</td>
<td>&gt; 0</td>
<td>unstable node</td>
</tr>
<tr>
<td>(1,0)</td>
<td>1</td>
<td>-2</td>
<td>&gt; 0</td>
<td>stable node</td>
</tr>
<tr>
<td>(0,-\sqrt{2})</td>
<td>&gt; 0</td>
<td>&lt; 0</td>
<td>&gt; 0</td>
<td>stable node</td>
</tr>
<tr>
<td>(0,+\sqrt{2})</td>
<td>&lt; 0</td>
<td>&lt; 0</td>
<td></td>
<td>saddle point</td>
</tr>
<tr>
<td>(1 + \sqrt{5}/4, -3 + \sqrt{5}/4)</td>
<td>&lt; 0</td>
<td>&lt; 0</td>
<td></td>
<td>saddle point</td>
</tr>
<tr>
<td>(1 - \sqrt{5}/4, -3 - \sqrt{5}/4)</td>
<td>&lt; 0</td>
<td>&lt; 0</td>
<td></td>
<td>saddle point</td>
</tr>
</tbody>
</table>

(b) x nullcline: \(y = 1 - x\)  
y nullcline: \(y = 2 - 3x\)  
These nullclines intersect at \((1/2,1/2)\). The Jacobian is

\[
J(x,y) = \begin{bmatrix}
-1 & -1 \\
-3 & -1
\end{bmatrix}
\]

We have \(det(J(1/2,1/2)) = -2 < 0\) and \(Tr(J(1/2,1/2)) = -2 < 0\). Hence, the equilibrium \((1/2,1/2)\) is a saddle point.

For those that did this question before it was updated, the nullclines are those already given together with the x-nullcline \(x = 0\) and the y-nullcline \(y = 0\). This yields equilibria \((1/2,1/2)\) (as above), \((0,0)\), \((0,2)\), and \((1,0)\). The Jacobian changes to,

\[
J(x,y) = \begin{bmatrix}
1 - 2x - y & -x \\
3y & 2 - 2y - 3x
\end{bmatrix}
\]

The stability of the equilibria are in the following chart:

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<tr>
<td>((1,0))</td>
<td>1</td>
<td>-2</td>
<td>= 0</td>
<td>stable node</td>
</tr>
<tr>
<td>((0,2))</td>
<td>2</td>
<td>-3</td>
<td>&gt; 0</td>
<td>stable node</td>
</tr>
</tbody>
</table>

(c) x nullcline: \(y = 0\)  
y nullcline: \(y = -\frac{1}{10}\sin(x)\)  
In the domain given, these intersect at \((0,0)\), \((\pi,0)\), and \((2\pi,0)\). The Jacobian is

\[
J(x,y) = \begin{bmatrix}
0 & 1 \\
-\cos(x) & -1/10
\end{bmatrix}
\]

So, \(det(J(x,y)) = \cos(x)\) and \(Tr(J(x,y)) = -1/10 < 0\). To check the equilibria, we need only put the \(x\) values into the determinant expression. For \((0,0)\), \(det(J(0,0)) = 1 > 0\), and so \((0,0)\) is a stable spiral. For \((2\pi,0)\), \(det(J(2\pi,0)) = 1\), and so \((2\pi,0)\) is also a stable spiral. For \((\pi,0)\), \(det(J(\pi,0)) = -1 < 0\), so \((\pi,0)\) is a saddle point. As it turns out, this system of equations represents a pendulum with damping. So cool!
(d) x nullclines: $x = \pm 1$

y nullclines: $x, y = 0$

These intersect at $(1, 0)$ and $(-1, 0)$. The Jacobian is

$$J(x, y) = \begin{bmatrix} 2x & 0 \\ -y & -x \end{bmatrix}$$

For $(-1, 0)$, $det(J(-1, 0)) = -2 < 0$, and $Tr(J(-1, 0)) = -1 < 0$. Hence, $(-1, 0)$ is a saddle point.

For $(1, 0)$, $det(J(1, 0)) = -2 < 0$, and $Tr(J(1, 0)) = 1 > 0$. Hence, $(1, 0)$ is a saddle point.

(e) x nullcline: $y = -x^2 + 4x + 2$

y nullcline: $x = 1$

These intersect at $(1, 5)$. The Jacobian is

$$J(x, y) = \begin{bmatrix} 2x - 4 & 1 \\ 1 & 0 \end{bmatrix}$$

We have $det(J(x, y)) = -1 < 0$ for any $(x, y)$. Hence, $(1, 5)$ is a saddle point.