DYNAMICAL SYSTEMS CHEAT SHEET WITH A SIDE OF SYPHILIS

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1-D Dynamical Systems

If you are presented with a single differential equation of the form:

\[ \frac{dx}{dt} = f(x) \]

Then you should do the following.

(1) **Find the equilibrium points:** Solve the equation \( f(x^*) = 0 \) for \( x^* \) (i.e., the values of \( x \) such that when you plug those points into \( f(x) \) you get zero.

(2) **Evaluate the stability of EACH equilibrium points:** To evaluate the stability, find \( f'(x) \) (the derivative of the right hand side of your differential equation). If

- \( f'(x^*) < 0 \) then \( x^* \) is a stable equilibrium point
- \( f'(x^*) > 0 \) then \( x^* \) is an unstable equilibrium point
- \( f'(x^*) = 0 \) then \( x^* \) hum... we can’t say anything definitive about \( x^* \)

(3) **Draw the Phase-diagram.** The phase diagram is a plot of \( \frac{dx}{dt} \) vs \( x \).

- Label the equilibrium points (the places where the line intersects the \( x \)-axis).
- Draw arrows on the \( x \)-axis around each equilibrium point. It the line (derivative) is above the \( y = 0 \) axis, then you draw an arrow in the positive (right) direction. If the line (derivative) is below the \( y = 0 \) axis, then you draw an arrow going to the left (negative direction).

2-D Dynamical Systems

Given two differential equations

\[ \frac{dx}{dt} = f(x, y) \]
\[ \frac{dy}{dt} = g(x, y) \]

You should do the following:
(1) **Find the equilibrium points** \((x^*, y^*)\) **via nullclines and draw the phase diagram:** The x-nullclines are found by setting the equation \(f(x, y) = 0\) and solving for \(x\) and \(y\). Along these lines (or curves) there is no change in the \(x\) direction. To solve for the y-nullclines, set the equation \(g(x, y) = 0\) and solve for \(x\) and \(y\). These lines (or curves) are the places on the graph where there there is no change in the \(y\)-direction. To find the **equilibrium points** locate on the phase diagram (plot of \(x\) vs \(y\)) where the x-nullclines intersect with the y-nullclines. This point of intersection corresponds to a point where there is no change in the \(x\)-direction and no change in the \(y\)-direction. (i.e., the system is at equilibrium = no change in any direction)

(2) **Draw direction arrows on your Phase-Diagram:** Draw arrows to indicate which way your system is moving. To do this, you can plug a coordinate \((x, y)\) into the \(f(x, y)\) and \(g(x, y)\) equations. If \(f(x, y) > 0\) at that point, you should draw an arrow to the right (the derivative of \(x\) is positive meaning it will be increasing in the \(x\)-direction), if \(f(x, y) < 0\) then draw an arrow going to the left (the derivative of \(x\) is negative i.e. decreasing at that point). If \(g(x, y) < 0\) draw an arrow pointing down and if \(g(x, y) > 0\) draw an arrow pointing up.

(3) **Calculate the Jacobian:** The Jacobian is a matrix of partial derivatives.

\[
J = \begin{bmatrix}
\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\
\frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}
\end{bmatrix}
\]

(4) **Evaluate the Jacobian at each fixed point and then Calculate the Determinant \(D\) and Trace \(T\):** After step (3). Plug an equilibrium point into \(J\). Now, \(J\) will be a 2x2 matrix of numbers (no more variables) like:

\[
J(x^*, y^*) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}
\]

\[
D = ad - bc \quad \text{and} \quad T = a + d
\]

(5) **Use \(D\) and \(T\) to determine the stability of each fixed point:** (see Table) At this point you will want to go back to your phase digram and draw in some trajectories around each equilibrium point. The little arrows you drew in step (2) should help. (ie. tell you which way you should be spiralling)

(6) **Step back and tell a story:** What have you learned about your dynamical system?
Table 1. Stability of an equilibrium point given D and T

| $D < 0$ | Saddle Point |
| $D > 0$ $T > 0$ | $(T^2 - 4D) \geq 0$ Unstable Node |
| $D > 0$ $T > 0$ | $(T^2 - 4D) < 0$ Unstable Spiral |
| $D > 0$ $T < 0$ | $(T^2 - 4D) < 0$ Stable Spiral |
| $D > 0$ $T < 0$ | $(T^2 - 4D) \geq 0$ Stable Node |
| $D > 0$ $T = 0$ | Center |

1. Syphilis Example

We talked about Syphilis in class. And you probably heard about it in grade 11 gym class. In short, it is an STI (sexually transmitted bacterial infection). Up until the 1950s it was a serious problem. Then came along penicillin. Penicillin was used to treat syphilis and the disease almost disappeared in the developed world. But then all of a sudden it came back again... and then it disappeared again... then came back... then disappeared... and since the 1950s we have been seeing these cycles. About every 10 years there is a syphilis outbreak. We will use what we know about 2-D dynamical systems to try to figure out why this is.

We can use the following two differential equations to model the spread of syphilis in a closed population (people never leave, never die and no new people show up). (We will go into details about how the model is derived in class)

\[ \frac{dS}{dt} = -\beta SI + \gamma (N - S - I) \]
\[ \frac{dI}{dt} = \beta SI - vI \]

- $\beta$ is the infection rate
- $v$ is the recovery rate from syphilis
- $\gamma$ is the rate at which people lose their resistance to syphilis
- $S$ is the susceptible category (or number of susceptible people)
- $I$ is the infected category (number of infected people)
- $N$ is the total population size (number of people in the population)

Let $\beta = 0.3$, $N = 500$, $\gamma = 0.8$, and $v = 25$.

**Step 1** The first thing we want to do is solve for the equilibrium points. To solve for the equilibrium points, we need to find the nullclines.

**x-nullclines** (aka S-nullclines)
First let’s solve for $I$

\[ f(x, y) = -\beta SI - \gamma (N - S - I) \]
\[ \beta SI = \gamma (N - S - I) \]
\[ I = \frac{\gamma (N - S)}{\beta S + \gamma} \]
and now we solve for $S$

\[
f(x, y) = 0 = -\beta SI - \gamma(N - S - I)
\]

\[
\beta SI = \gamma(N - S - I)
\]

\[
S = \frac{\gamma(N - I)}{\beta I + \gamma}
\]

Woo-Wee-WoW... Turns out $S$ and $I$ are the EXACT same line (curve). If you don’t believe me, take the $I$ equation and solve for $S$ and you will see. So, we have only ONE S-nullcline.

**y-nullclines** (aka I-nullclines)

\[
g(x, y) = 0 = \beta SI - vI
\]

\[
0 = I(\beta S - v)
\]

\[
I = 0
\]

\[
S = \frac{v}{\beta}
\]

So, there are 2 y-nullclines.

Let's do some plotting.

There are two places on the graph where S-nullclines intersect with I-nullclines. These two points are $(83,13)$ and $(500,0)$. These are the two equilibrium points of this system.

**Step 2** Yah.. I am going to cheat and use maple.
Step 3 To determine the stability of each equilibrium point we need to calculate the Jacobian of our 2-D system.

\[
J = \begin{bmatrix}
\frac{\partial f}{\partial S} & \frac{\partial f}{\partial I} \\
\frac{\partial g}{\partial S} & \frac{\partial g}{\partial I}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
-\beta I - \gamma & -\beta S - \gamma \\
\beta I & \beta S - \nu
\end{bmatrix}
\]

\[
= \begin{bmatrix}
-0.3I - 0.8 & -0.3S - 0.8 \\
0.3I & 0.3S - 25
\end{bmatrix}
\]

Step 4 and 5 Evaluate the Jacobian at each equilibrium point.

First let's look at \((S^*, I^*) = (500, 0)\)

\[
J(500,0) = \begin{bmatrix}
-0.3(0) + 0.8 & -0.3(500) + 0.8 \\
0.3(0) & 0.3(500) - 25
\end{bmatrix}
\]

\[
= \begin{bmatrix}
-0.8 & -150.8 \\
0 & 125
\end{bmatrix}
\]

\(D = -100 < 0\) According to the chart we have a SADDLE POINT

Now let's look at \((S^*, I^*) = (83, 13)\)
\[ J(83,13) = \begin{bmatrix} -0.3(13) - 0.8 & -0.3(83) - 0.8 \\ 0.3(13) & 0.3(83) - 25 \end{bmatrix} \]

\[ = \begin{bmatrix} -4.7 & -25.7 \\ 3.9 & -0.1 \end{bmatrix} \]

\[ D = 100.7 > 0 , T = -4.8 < 0 , (T^2 - 4D) = (-4.8)^2 - 4(94.46) = -379.7 < 0 . \]

According to the table we have a **STABLE SPIRAL** At this point we should go back to the phase portrait (diagram) and fill in the extra details. (check out throne at the bottom of this pdf)

**Step 6 Lets tell a story.** We have a simple syphilis model that makes certain assumptions about the population and how the disease spreads. But, we can still get some interesting information out of it. According to our model, \((500,0)\) is not a stable state. It is a bit of a trivial equilibrium point. If there are no infected people, there is no epidemic and thats that. But what if we have only one infected person? Because \((500,0)\) is a saddle point, that would mean that the disease would spread (move away from that point) and we would likely have an epidemic. Now consider the point \((83,13)\). According to our analysis, this point is a stable spiral. This could explain why we have been seeing cycling syphilis outbreaks in the data. Because of the relatively short time one spends in the infectious category compared to the time spent in the resistant or susceptible categories, busts of infections are followed by long periods of resistance. Perhaps it is during this period that the high risk individuals who would usually contract the disease are immune and there is no one to spread the disease. Eventually these high risk individuals lose their immunity and become infected once again. This is a very simplified model and small alterations in it could lead to much different dynamics. Even small changes in parameter values can change the stability of the system. If we were to make the infection period longer, the equilibrium point would go from stable spiral to saddle.

Below is the solution to our syphilis system with initial condition \((100,20)\) 100 susceptible people and 20 infectious people. Notice how both the infectious and susceptible populations cycle up and down before settling down to the equilibrium value.
Phase Diagram for Syphilis