Name:                              Student #: 

Instructions:

• Write your name and student number in the spaces provided. Midterms missing a name or student number will be penalized five percent. Midterms missing both items will receive a mark of zero.

• Answer the questions in the spaces provided to the best of your ability. Your answers will be graded on, in order of ascending importance, brevity, clarity and correctness. If your answers fail to satisfy these criteria it will result in loss of marks.

• You may use the back of the pages to continue your answers if there is not enough room, but be sure to clearly indicate where your solution is if you do so.

• The use of a calculator is permitted. All other devices or aids, electronic or otherwise, are prohibited. If you are wearing any kind of smart device, please remove it and put it away for the duration of the test.

• If you have a question, raise your hand and wait for the proctor to come to you.

• You have two hours to complete this test. There are 2 sections, the first has ten questions, the second has 5 questions. The test is out of a total of 70 marks. Good luck.
Section 1 (20 marks) Short Answer: Give the definitions of the following terms, add the elements that are missing to a definition, or answer the problem in one or two steps. Your definitions do not need to be formal, but a definition that is not clear, or correct, will be penalized marks.

1. (4 marks) Give a two sentence description describing the tragedy of the commons type game. Your first sentence should be about the Nash equilibrium. Your second sentence should be about the payoffs to the group depending on their choices.

*TOTC type games have a Nash equilibrium where everyone chooses the move with the higher payoff at the expense of others. When everyone selects this choice, everyone’s payoff is lower than it would be if they all had chosen the move with the lower payoff.*

2. (2 marks) Complete the definition: A Nash equilibrium for an n-player game is a strategy profile \((s^*_1, s^*_2, ..., s^*_n)\) with the property:

*No player can increase her payoff by changing strategy when everyone else is playing at the NE.*

3. (1 mark) True or False: If you use IEDS on a payoff matrix where all of its strategies are either strictly dominant or dominated, the order of elimination of rows and columns does not matter, although the NE you arrive at may not be unique.

*False*

4. (1 mark) Complete the definition: An active pure strategy of player \(i\) is an \(s_i \in S_i\) has the property that

*The probability that a player uses \(s_i\) is greater than zero.*
5. (2 marks) State the Existence Theorem of Nash Equilibria.
If, in an n-person game in normal form, each player’s strategy set is finite, then the game has at least one mixed strategy Nash equilibrium.

6. (3 marks) Consider a two-player zero sum game. How do you find the saddle point of the payoff matrix for that game? Assuming there is one, classify the game based on the saddle point’s value in terms of fairness.
   The saddle point is the minimum of its row and the maximum of its column in the payoff matrix. If the SP = 0, the game is fair, if SP > 0, the game favours P1, if SP < 0, the game favours P2.

7. (1 marks) Complete the following definition: The win-loss pattern for the generalized Subtraction game is known as:
The Nim-sequence.

9. (3 marks) What is the Nim-sum of 14, 22 and 31, in base 10?
7.

10. (3 marks) State Bouton’s Theorem for Nim.
A game state \((x_1, x_2, ..., x_n)\), where \(x_i \geq 0, i = 1, 2, ..., n\), is a losing state in Nim if and only if the Nim-sum of its piles is 0.
Section 2 (50 marks) Solve the following problems. Answers can be in written sentences and/or equations.

1. (10 marks) Consider a two-player zero sum game with the following payoff matrix:

\[
\begin{pmatrix}
1 & 4 & 10 \\
2 & 5 & 1 \\
3 & 6 & 9 \\
\end{pmatrix}
\]

Start by reducing the game to a 2 by 2 matrix \( H \). Now, without using IEDS or finding the saddle point, prove that \( H \) has a single dominant strategy.

Reduction the matrix to \( H = \begin{pmatrix} 1 & 10 \\ 3 & 9 \end{pmatrix} \) is worth 4 marks. The proof is as follows: begin by assuming \( H \) does not have a single dominant strategy. Then we can use the MSNE procedure to find a mixed strategy for the NE. 2x2 TPSZG have already been solved, and we can use the formula

\[
p = \frac{d - c}{a - b + d - c}
\]

to find the probability that Player 1 uses the move corresponding to row 1. A quick calculation shows that \( p = -2 \), which is impossible, since \( 0 \leq p \leq 1 \). Therefore we can conclude that \( H \) does not have a MSNE, and must have a single dominant strategy instead. \( \square \)
2. (10 marks) Are there any 2 by 2 TPZSG that exist that have saddle points which cannot be found by IEDS? Prove your answer, either way.

There are no matrices that have saddle points that cannot be found by IEDS. Start with the general matrix (3 marks for starting with a general matrix, examples are worth 1 point total, not each)

\[
\begin{pmatrix}
  a & b \\
  c & d \\
\end{pmatrix}
\]

Without loss of generality, let \( a \) be the saddle point, which implies \( a > c, a < b \rightarrow b > c \) (3 marks for this statement somewhere). Now we need to consider how \( b \) is related to \( d \) and how \( c \) is related to \( d \). If \( b > d \rightarrow \) Row 1 dominates Row 2. This means that Column 1 dominates Column 2, and we arrive at \( a \) by IEDS (1 mark).

If \( c > d \rightarrow b > d \rightarrow \) IEDS will find \( a \) as a solution, by similar reasoning above (1 mark).

If \( c < d \) and \( b < d \rightarrow \) Column 1 dominates Column 2, which gives \( a \) as the solution by IEDS (1 mark).

Therefore there are no 2x2 TPZSG matrices that have saddle points that cannot be found by IEDS (1 mark for some kind of coherent sentence at the end).
3. (10 marks) Consider the game state (17, 25, 30, 2) in the game of Nim. Is this a winning state for Player 1, or a winning state for Player 2? Find 3 moves that would be considered optimal play.

Start with the Nim-sum

\[
\begin{array}{cccccc}
1 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 1 \\
+ & 1 & 1 & 1 & 1 & 0 \\
\hline
1 & 0 & 1 & 1 & 0 \\
\end{array}
\]

The Nim-sum is 22 in base 10, therefore this is a winning state for Player 2 (6 marks for this, give 3 if they make a mistake somewhere).
The 3 optimal moves are take 22 from 30, leaving 8. Take 10 from 17 leaving 7. Take 10 from 25 leaving 15. (2 marks for each one, all or nothing).
4. (10 marks) Consider the following payoff matrix:

\[
\begin{pmatrix}
(9, -4) & (4, -2) & (-2, -3) \\
(-2, 4) & (5, 3) & (0, 2) \\
(5, 2) & (2, 4) & (1, 3)
\end{pmatrix}
\]

Find the MSNE and the expected payoffs for both players. Which player does this game favour? (As a reminder, you get half marks for process, and half marks for the correct answer. Take your time on this one.)

This matrix can be reduced to

\[
\begin{pmatrix}
(9, -4) & (4, -2) \\
(-2, 4) & (5, 3)
\end{pmatrix}
\]

(2 marks for this step.) Using the MSNE algorithm:

\[
\pi_2(\sigma_1, C) = -4p_A + 4(1 - p_A), \pi_2(\sigma_1, D) = -2p_A + 3(1 - p_A)
\]

\[
p_A = \frac{1}{3}, p_B = \frac{2}{3}
\]

\[
\pi_1(A, \sigma_2) = 9p_C + 4(1 - p_C), \pi_1(B, \sigma_2) = -2p_C + 5(1 - p_C)
\]

\[
p_C = \frac{1}{12}, p_D = \frac{11}{12}
\]

The expected payoff is

\[
E(\pi_1) = \frac{1}{3}(\frac{1}{12} 9 + \frac{11}{12} 4) + \frac{2}{3}(\frac{1}{12} -2 + \frac{11}{12} 5) = \frac{53}{36} \approx 4.4167
\]

\[
E(\pi_2) = \frac{1}{3}(\frac{1}{12} -4 + \frac{11}{12} -2) + \frac{2}{3}(\frac{1}{12} 4 + \frac{11}{12} 3) = \frac{4}{3} \approx 1.3333
\]

(7 marks for finding both of these values). This game favours player 1, since their expected payoff is higher at the NE (1 mark for this sentence).
5. (10 marks) Recall the game Cliquer. A player gets one point for drawing an edge between two vertices, three more points if that edge creates a clique of size 3, and five more points if that edge creates a clique of size four, as long as the graph remains planar. Consider the following game state (vertices are numbered for ease of analysis):

If it is your turn, find a move that maximizes the number of points you can get in this game state, and show that move guarantees the highest number of points your opponent can get on their turn afterwards is four.

There are 6 moves you can make at this point in the game: 2-5, 2-6, 5-6, 3-6, 1-6, and 1-4. There are two moves that get you twenty points, 1-4 and 2-5. If you choose 1-4, you create $K_3$'s out of 1, 2 and 4, 1, 4, and 5, and 1, 3 and 4. You create $K_4$'s out of 1, 2, 3 and 4, and 1, 3, 4, and 5. If you choose 2-5, you create a $K_3$ out of vertices 2, 4, and 5, another $K_3$ out of 1, 2, and 5, and one more out of 2, 3, and 5. You also create 2 $K_4$'s, 2, 3, 4, and 5, and 1, 2, 3, and 5, gaining you a total of 20 points (1 for the edge, 9 for the $K_3$'s and 10 for the $K_4$'s. (5 points for finding either move). The other moves give you 1 or 4 points. If you draw 1-4, you can block off 5 from 2 and 6, assuming you draw your arc around the “right” side of the graph. This means that there is only one move left for the next player, 2-6, which is worth 4 points. If you draw 2-5 and go “underneath” the graph, you are blocking off 6 from 1, and the only move left is 5-6, which is also worth 4. (5 points for proving maximum score is 4) □