Chapter 3

Games in Normal Form and the Nash Equilibrium

Introduction and Definitions

Games in extensive form with complete information can be used to think about many situations and help us analyze how to approach a given decision. However, GEFCI are limited because they can never represent a situation in which both players make their decisions at the same time. Unlike the games in extensive form with complete information, in which players take turns making moves, a game in normal form (GNF) consists of players making decisions simultaneously. A GNF

- Has a finite set of players $\mathbb{P} = \{1, ..., n\}$
- For each $i \in \mathbb{P}$, $S_i$ is the set of available strategies to each player.

Note: We can define the strategy of a player $p$, $S_p$, as the move $p$ will make for every situation they encounter in the game. The size of $S_p$, $|S_p|$ is given by the available choices to $p$ in the game. Using this definition, we can also define $\mathbb{S} = S_1 \times S_2 \times ... \times S_n$ as the complete space of strategies for a given $n$ player game.

- An element of $\mathbb{S}$ is an $n$-tuple $(s_1, ..., s_n)$, where each $s_i$ is a strategy chosen from $S_i$. 
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- The n-tuple \((s_1, ..., s_n)\) is called a strategy profile, and represents the choice of strategy of each the players participating in the game.

- For each \(i \in \mathbb{P}\), there exists a payoff function \(\pi_i : S \rightarrow \mathbb{R}\).

Consider the game with 2 players. If player 1 has \(m\) strategies and player 2 has \(n\) strategies, then the game in normal form can be represented as an \(m \times n\) matrix of ordered pairs of payoffs. This chapter has a few examples to illustrate the concepts being described, try to describe the set of players, their available strategies, and a couple of strategy profiles for each example. This will help you keep the technical parts of this chapter understandable.

The Stag Hunt

The story of the Stag Hunt, initially told by Jean-Jacques Rousseau, in *A Discourse on Inequality* in the 18th century, is as follows:

“If it was a matter of hunting a deer, everyone well realized that he must remain faithful to his post; but if a hare happened to pass within reach of one of them, we cannot doubt that he would have gone off in pursuit of it without scruple...""}

Another way to think of this is two hunters go out and agree to occupy opposite sides of a game track. The hunters can’t see each other, but when large prey comes by, such as a stag, one hunter can attempt to kill it. If the stag were to run away, it would run into the other hunter waiting for it, thus increasing the chances that the two hunters could bring the stag down. Hunting a stag is good, likely to provide for the hunters and their families for a while. However, if one of the hunters abandons their post to hunt down easier prey, say a hare, that hunter probably will get the hare and eat for a day, but they leave the other hunter without help, making it unlikely that the stag will be brought down. The other hunter will go hungry. Unlike the the Prisoner’s Dilemma, the Stag Hunt does not have a sucker or temptation payoff. The payoff of defecting (hunting hare) if your opponent cooperates (hunting stag) is not better than if you both hunted stag. David Hume and Thomas Hobbes also have discussed their own versions of this concept, but the man in recent history to have really nailed down the Stag Hunt is Brian Skyrms. His book *The Stag Hunt and Evolution of Social Structure* is an excellent read, and he has since wrote several papers on the
topic since the release of the book. The Prisoner’s Dilemma has been used as a model for social interaction for many years, but Skyrms, among others, has given strong arguments that the Stag Hunt is better at representing many of those social situations. We can construct the Stag Hunt as a game in normal form with the following conditions:

- There are two players.
- Each player has the available strategy hunt stag or hunt hare.
- The payoffs for both players are given in the following matrix:

<table>
<thead>
<tr>
<th></th>
<th>HS</th>
<th>HH</th>
</tr>
</thead>
<tbody>
<tr>
<td>Player 1</td>
<td>HS</td>
<td>(5,5)</td>
</tr>
<tr>
<td></td>
<td>HH</td>
<td>(2,0)</td>
</tr>
</tbody>
</table>

Similar to games in extensive form, payoffs are listed as ordered pairs (or n-tuples if there are more two players), with the first term being the payoff to Player 1, the second to Player 2, and so on. So in our version of the Stag Hunt, if both players hunt stag, they receive a payoff of 5. We can generalize the Stag Hunt by creating the payoff matrix:

<table>
<thead>
<tr>
<th></th>
<th>HS</th>
<th>HH</th>
</tr>
</thead>
<tbody>
<tr>
<td>Player 1</td>
<td>HS</td>
<td>(a,a)</td>
</tr>
<tr>
<td></td>
<td>HH</td>
<td>(b,c)</td>
</tr>
</tbody>
</table>

If $a \geq b \geq d \geq c$, then this is called a Generalized Stag Hunt. If the conditions that $2a \geq b+c$ and $b \geq a \geq d \geq c$ are satisfied, this payoff matrix represents a Prisoner’s Dilemma.

**Dominated Strategies**

For a game in normal form, let $s_i$ and $s_j$ be two of player $p$’s strategies ($s_i, s_j \in S_p$).

- We say $s_i$ strictly dominates $s_j$ if, for every choice of strategies by the other n-1 players of the game, $\pi_p(s_i) > \pi_p(s_j)$.
- We say $s_i$ weakly dominates $s_j$ if, for every choice of strategies by the other n-1 players of the game, $\pi_p(s_i) \geq \pi_p(s_j)$.
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The game in normal form still assumes rationality of all players and this implies that if player $p$’s strategy $s_i$ weakly (or strictly) dominates $s_j$, then $p$ will not choose $s_j$.

Iterated Elimination of Dominated Strategies

Recall that backward induction helped us solve games in extensive form with complete information by eliminating moves that yielded lower payoffs. The removal of strategies that yielded lower payoffs reduced the size of the game, assuming that different moves yielded unique payoffs from a given node. Games in normal form follow a similar process. The assumption of rationality of players implies that dominated strategies will not be used by a player. Thus, we can remove a dominated strategy from a game in normal form, and we are often left with a smaller game to analyze.

If the game reduces down to one strategy for a given player, we call that the dominant strategy. If there is a single dominant strategy for every player in the game, the strategy profile $(s^*_1, ..., s^*_n)$ is called a dominant strategy equilibrium.

Fact: Iterated elimination of strictly dominated strategies (IEDS) produces the same reduced game, regardless of the order in which dominated strategies are removed from the game. However, order of elimination can matter when removing weakly dominated strategies.

3.1 The Nacho Game

The Nacho Game was created by Brandon Chai and Vincent Dragnea. As they put it, this game was born at the pub, over a plate of nachos. When you share a plate of nachos with a friend, you have the option of eating nachos at an enjoyable, slower pace, or eating the nachos at a quicker, less enjoyable pace that ensures you eat more of them than your friend. This can be modeled as a two player game in normal form.

- For the ease of analysis, we limit the speed at which a player can eat nachos: slow and fast.
- At slow speed, a player eats 5 nachos per minute, and gets a payoff of 1 for each nacho eaten.
3.1. THE NACHO GAME

- At fast speed, a player eats 10 nachos per minute, and gets a payoff of \( \frac{4}{5} \) for each nacho eaten.
- Both players make one decision, the speed at which to eat the nachos.
- A plate of nachos has \( n \) nachos to be consumed at the start of the game.

The payoffs for the Nacho Game are as follows:

<table>
<thead>
<tr>
<th>Player 2 (Eat Slow, Eat Fast)</th>
<th>Player 1 (Eat Slow)</th>
<th>Player 1 (Eat Fast)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \left( \frac{n}{2}, \frac{n}{2} \right) )</td>
<td>( \left( \frac{n}{3}, \frac{8n}{15} \right) )</td>
<td>( \left( \frac{2n}{3}, \frac{2n}{5} \right) )</td>
</tr>
<tr>
<td>( \left( \frac{8n}{25}, \frac{n}{5} \right) )</td>
<td>( \left( \frac{n}{4}, \frac{2n}{5} \right) )</td>
<td>( \left( \frac{n}{3}, \frac{3n}{3} \right) )</td>
</tr>
</tbody>
</table>

If we examine the payoffs, this game satisfies the conditions of a Prisoner’s Dilemma! In terms of dominated strategies, we can see that choosing to eat nachos at a fast pace (defect) always provides a larger payoff than choosing to eat nachos at a slow pace (cooperate), regardless of what decision the other player makes. Thus, the solution to the Prisoner’s Dilemma, based on IEDS, is that both players should defect.

3.1.1 The Nacho Game with K Players

Unlike the Prisoner’s Dilemma (not including its many, many variants), the Nacho Game allows for more than two players to participate. Indeed, picture a round table full of hungry undergraduate students staring at a plate of fresh, hot, cheese covered nachos, along with all your favourite toppings. We will start by analyzing the game with three players, and generalize from there. A modified payoff matrix is necessary here.

This modified payoff matrix shows the payoffs to all three players. The rows are the choices made by Player 1 (without loss of generality, this could be Player 2 or 3). The columns represent the choices made by the other players, and the ordered triplets are the respective payoffs. If we use IEDS, once again we can see that it is in the best interest of Player 1 to eat fast, regardless of what choices the other players make.

We can generalize, and reduce, this payoff matrix for a single player regardless of how many players participate in the Nacho Game. Let \( k \) be the number of players, and let \( h \) be the number of players - not including yourself - that decide to eat fast. Then the payoffs to Player 1 are...
### 6. Chapter 3. Games in Normal Form and the Nash Equilibrium

<table>
<thead>
<tr>
<th>Payoff</th>
<th>Fast</th>
<th>( \frac{2n}{k+h+1} \cdot \frac{4}{5} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Slow</td>
<td>( \frac{n}{k+h} )</td>
<td></td>
</tr>
</tbody>
</table>

Is it ever more profitable to eat slowly? We can check the conditions under which the slow payoff is greater than the fast payoff.

\[
\frac{n}{k+h} > \frac{2n}{k+h+1} \cdot \frac{4}{5} \tag{3.1}
\]
\[
\frac{5}{k+h} > \frac{8}{k+h+1} \tag{3.2}
\]
\[
5k+5h+5 > 8k+8h \tag{3.3}
\]
\[
5 > 3(k+h) \tag{3.4}
\]

If \( k = 1 \), the equation is true. So it is better to eat slow when you are eating a plate of nachos by yourself (no comment on the sadness of this endeavour). However, if \( k > 1 \rightarrow h \geq 0 \), 3\((k+h)\) is never less than 5, so under no conditions is the slow payoff greater than the fast payoff with more than 1 person playing. Thus, we have shown that the eat fast strategy strictly dominates the eat slow strategy.

### 3.2 Nash Equilibria

Consider a game in normal form with \( n \) players, with strategy sets \( S_1, S_2, ..., S_n \) and payoffs \( \pi_1, \pi_2, ..., \pi_n \). A **Nash equilibrium** is a strategy profile \( (s_1^*, s_2^*, ..., s_n^*) \) with the following property: if any player changes his strategy, his own payoff will not increase. The strategy profile \( (s_1^*, s_2^*, ..., s_n^*) \) is a Nash equilibrium if:

- For all \( s_1 \in S_1, \pi_1(s_1^*, s_2^*, ..., s_n^*) \geq \pi_1(s_1, s_2^*, ..., s_n^*) \)
- For all \( s_2 \in S_2, \pi_2(s_1^*, s_2^*, ..., s_n^*) \geq \pi_2(s_1^*, s_2, ..., s_n^*) \)
  ...
- For all \( s_n \in S_n, \pi_n(s_1^*, s_2^*, ..., s_n^*) \geq \pi_n(s_1^*, s_2^*, ..., s_n) \)

The strategy profile \( (s_1^*, s_2^*, ..., s_n^*) \) is a **strict** Nash equilibrium if:

- For all \( s_1 \neq s_1^*, \pi_1(s_1^*, s_2^*, ..., s_n^*) > \pi_1(s_1, s_2^*, ..., s_n^*) \)
- For all \( s_2 \neq s_2^*, \pi_2(s_1^*, s_2^*, ..., s_n^*) > \pi_2(s_1^*, s_2, ..., s_n^*) \)
  ...

3.2. NASH EQUILIBRIA

- For all \( s_n \neq s_n^* \), \( \pi_n(s_1^*, s_2^*, ..., s_n^*) > \pi_n(s_1^*, s_2^*, ..., s_n) \)

The Nash equilibrium (NE) is one of the most important concepts in classical game theory. You will notice that we have been finding the NE in every game I have introduced, although I have not specified it as such. If you go back and look over the Nacho Game, the solution to that game is the NE, as it has been defined above. The NE of the Prisoner’s Dilemma is Defect, and the Stag Hunt has two NE, (HS,HS) and (HH,HH). From now on when analyzing games we will be searching for the NE when appropriate.

3.2.1 Finding the NE by IEDS

The following two theorems demonstrate the utility of IEDS for finding NE. They are offered without proof.

**Theorem 1** Suppose we do iterated elimination of weakly dominated strategies on a game \( G \), where \( G \) is a GNF. Let \( H \) be the reduced game. Then:

1. Each NE of \( H \) is also a NE of \( G \).

2. If \( H \) has one strategy, \( s_i^* \) for each player, then the strategy profile \((s_1^*, s_2^*, ..., s_n^*)\) is a NE of \( G \).

**Theorem 2** Suppose we do IEDS, where the strategies are strictly dominated in a game \( G \), \( G \) is a GNF. Then:

1. Strategies can be eliminated in any order, it will result in the same reduced game \( H \).

2. Each eliminated strategy is not part of the NE of \( G \).

3. Each NE of \( H \) is a NE of \( G \).

4. If \( H \) only has \( s_i^* \) for each player, then the strategy profile \((s_1^*, s_2^*, ..., s_n^*)\) is a strict NE of \( G \), and there are no other NE.
3.2.2 IEDS process

The IEDS process for two players is actually fairly simple, and can easily be
generalized to a game with n players. Considering the two player game, the
easiest way to remember how to eliminate dominated strategies is with the
following two ideas:

- If Player 1 has a dominated strategy by some other strategy, that will
  be shown by comparing the first element of each pair in two rows. Each
  element of the dominant strategy will be greater than each element of
  the dominated strategy, and you can eliminate that entire row from the
  matrix.

- If Player 2 has a dominated strategy by some other strategy, that will
  be shown by comparing the second element of each pair in two columns.
  Each element of the dominant strategy will be greater than each element
  of the dominated strategy, and you can eliminate that entire column from
  the matrix.

- Bottom line: you eliminate rows by comparing first terms in the pairs,
  and you eliminate columns by comparing second terms. The order of
  elimination is irrelevant when finding Nash equilibria by iterated elimi-
  nation of strictly dominated strategies, but it can matter when some
  or all of the eliminated strategies are weakly dominated.

For example,

\[
G = \begin{pmatrix}
(15, 2) & (0, 3) & (8, 2) \\
(4, 1) & (4, 4) & (7, 3) \\
(-2, 8) & (3, 9) & (7, -2)
\end{pmatrix}
\]

Take a look at the middle and right columns of G. Notice that 3 > 2, 4 > 3, 9 > -2, so the middle column dominates the right column, and we can
delete the right column, getting

\[
H = \begin{pmatrix}
(15, 2) & (0, 3) \\
(4, 1) & (4, 4) \\
(-2, 8) & (3, 9)
\end{pmatrix}
\]

Now the middle and bottom row of H can be compared, and since 4 > -2, 4 > 3, we can see that the middle row dominates the bottom, so we
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delete the bottom row.

\[
H = \begin{pmatrix}
(15, 2) & (0, 3) \\
(4, 1) & (4, 4)
\end{pmatrix}
\]

If we compare the left and right column, \(3 > 2, 4 > 1\), thus we delete the left column.

\[
H = \begin{pmatrix}
(0, 3) \\
(4, 4)
\end{pmatrix}
\]

And finally, the bottom row dominates the top row, so the Nash equilibrium of this game is the strategies which give the payoffs \((4,4)\).

3.3 The Vaccination Game

The Vaccination Game, created by [?], is an excellent combination of game theory and mathematical biology. If a group of people face the possibility of catching an infectious disease and have the option getting vaccinated with a completely effective vaccine, how many will choose to be vaccinated? The authors use a static model where individuals have to choose whether or not to be vaccinated. The group of people in the model consider the costs and benefits of those decisions by making assumptions about who else will get vaccinated. Those costs can be thought of as monetary, psychological, and health related (perhaps as an adverse reaction to the vaccine). The following definitions will be necessary to discuss this model.

- Let \(c_i\) be the cost to Player \(i\) to get vaccinated.
- Let \(L_i\) be the loss to \(i\) if they catch the disease. We assume that an individual can still catch the disease from the background even if they do not encounter another person for the duration of the disease.
- Define \(p_i\) to be the probability of catching the disease even if no one else has it.
- Define \(r\) to be the probability that an infected person will infect someone who is not vaccinated, thus \(rp_i\) is the chance of catching the disease from a non-human source and then infecting another susceptible person. Denote \(rp_i\) as \(q_i\).
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- Assume a person who is vaccinated cannot transfer the disease.
- \( Y_i \) is \( i \)'s initial income or welfare.

Let us consider the two person case, where \( V \) is the choice to vaccinate, and \( NV \) is the choice to not vaccinate.

\[
\begin{array}{c|cc}
  & V & NV \\
\hline
V & Y_1 - c_1, Y_2 - c_2 & Y_1 - c_1, Y_2 - p_2L_2 \\
NV & Y_1 - p_1L_1, Y_2 - c_2 & Y_1 - p_1L_1 - (1 - p_1)q_2L_1, Y_2 - p_2L_2 - (1 - p_2)q_1L_2 \\
\end{array}
\]

From this payoff matrix, we can do the following analysis:

1. When \( c_i < p_iL_i \), for \( i = 1, 2 \), then \((V,V)\) is a Nash equilibrium, as these strategies dominate all others.

2. For \( p_1L_1 < c_1 \) and \( c_2 < p_2L_2 + (1 - p_2)q_1L_2 \), \((NV,V)\) is a NE, and \((V,NV)\) is an equilibrium if 1 and 2 are exchanged in the inequalities.

3. For \( p_1 < c_i < p_iL_i + (1 - p_i)q_jL_i \) for \( i, j = 1, 2 \), then both \((NV,V)\) and \((V,NV)\) are NE.

4. For \( p_1L_i + (1 - p_i)q_jL_i < c_i \), for \( i, j = 1, 2 \), then \((NV, NV)\) is the NE.

Note: Another paper, The Theory of Vaccination and Games, by [?], also uses game theory to analyze vaccination strategies.

3.3.1 The N-player Vaccination Game

Consider the game with \( N \) people rather than just two. Define \( R_i(K) \) to be the risk to individual \( i \) of infection when she is not vaccinated and those agents in the set \( \{K\} \) are vaccinated. With this notation, let’s restate the results of the two person vaccination game:

1. When \( c_i < R_i(j)L_i \), for \( i = 1, 2 \), then \((V,V)\) is a Nash equilibrium.

2. For \( R_1(2)L_1 < c_1 \) and \( c_2 < R_2(\emptyset)L_2 \), \((NV,V)\) is a NE, and \((V,NV)\) is an equilibrium if 1 and 2 are exchanged in the inequalities.
3. For $R_i(j)L_i < c_i < R_i(\emptyset)L_i$ for $i,j = 1,2$, then both $(NV,V)$ and $(V,NV)$ are NE.

4. For $R_i(\emptyset)L_i < c_i$, for $i,j = 1,2$, then $(NV,NV)$ is the NE.

This suggests that there is a general pattern, given in the proposition below.

**Proposition 1** Let there be $N$ people exposed to an infectious disease, with a probability $r$ of catching this from an infected person and a probability $p$ of catching it from a non-human host. $c$ is the cost of vaccination and $L$ is the loss from catching the disease. $R(\emptyset) = R(0)$ is the probability of a non-vaccinated person catching the disease if no one is vaccinated, and $R(k)$ is the probability of a non-vaccinated person catching the disease if $k$ are vaccinated. Then at the Nash equilibria the number of people vaccinated is as follows: for $R(j)L < c < R(j-1)L$ there are $j$ people vaccinated, $N - j$ not vaccinated. For $c < R(N-1)L$, everyone is vaccinated, and for $R(0)L < c$, no one is vaccinated.

The proof of this proposition can be found in the paper by Heal and Kunreather.

### 3.4 Exercises

1. Consider the Vaccination Game, and let’s impose a structure on the population of $N$ people. A small world network is one in which the population is divided into small clusters of highly connected people, like a family, or a classroom of young students, or people who work in the same office every day. These clusters of people are weakly connected to other clusters. So a family might be connected to another family down the street by virtue of the fact of the friendship between a single child from each family. Or two offices are connected by the custodian that cleans each office at the end of the day. There are many varieties of this sort of small world network. Picture a small world network on $N$ people, where there are $n$ small worlds, and on average each small world contains $\frac{N}{n}$ people, give or take a few. In each small world, everyone is completely connected to everyone else, or near enough that it makes no difference. Each small world is connected to at least one other small world by exactly one person. A small world may be connected to more than one small world, but each connection is through a unique individual, so one person in a small world can only be connected to one
other person in another small world, and can’t be connected to anyone else in other small worlds. Modify Proposition 1 such that it takes the small world network into account, and the policy of being able to force at least one person in each world to vaccinate, and prove that it is the optimal strategy. Recall that everyone chooses to vaccinate, or not to vaccinate, at the same time.

2. The tragedy of the commons The Nacho Game is an example of a class of situations called the tragedy of the commons. Here is another version. There are \( s \) students who have to do an assignment question. The students know that the professor is likely to not notice that the students have copied at least a few of their answers from each other, or the Internet (and shame on those students). In fact, the students know that the professor is likely to let a total \( as \) of these questions get by. The \( ith \) student has a choice of two strategies:

(a) The responsible strategy: cheat on \( a \) answers.

(b) The irresponsible strategy: cheat on \( a + 1 \) answers.

Each answer that is gained by cheating gives a payoff of \( p > 0 \) to the student. However, each student who cheats on \( a + 1 \) answers imposes a cost of getting caught \( c > 0 \) on the community of students, because of the professor will start to notice and dish out penalties accordingly. The cost is shared equally by the \( s \) students, because the professor can’t know who cheated more or less than others. Assume \( \frac{c}{s} < p < c \). Thus the cost of cheating on one more answer is greater than the profit from the cheating, but each student’s share of the cost is less than the profit.

(a) Show that for each student, cheating on \( a + 1 \) questions strictly dominates cheating on \( a \) answers.

(b) Which of the following gives a higher payoff to each student? (i) Every student cheats on \( a+1 \) questions or (ii) Every student cheats on \( a \) answers. Give the payoffs in each case.

3. Iterated Elimination of Dominated Strategies Use iterated elimination of dominated strategies to reduce the following games to smaller games in which iterated elimination of strictly dominated strategies
cannot be used further. State the order in which you eliminate strategies. If you find a dominant strategy equilibrium, say what it is. Remember, you eliminate rows by comparing first entries in the two rows, and you eliminate columns by comparing second entries in the two columns.

(a) \[
\begin{pmatrix}
 s_1 & (73, 25) & (57, 42) & (66, 32) \\
 s_2 & (80, 26) & (35, 12) & (32, 54) \\
 s_3 & (28, 27) & (63, 31) & (54, 29)
\end{pmatrix}
\]

(b) \[
\begin{pmatrix}
 s_1 & (63, -1) & (28, -1) & (-2, 0) & (-2, 45) & (-3, 19) \\
 s_2 & (32, 1) & (2, 2) & (2, 5) & (33, 0) & (2, 3) \\
 s_3 & (56, 2) & (100, -6) & (0, 2) & (4, -1) & (0, 4) \\
 s_4 & (1, -33) & (-3, 43) & (-1, 39) & (1, -12) & (-1, 17) \\
 s_5 & (-22, 0) & (1, -13) & (-2, 90) & (-2, -57) & (-4, 73)
\end{pmatrix}
\]

4. Find all two by two matrix games that do not have a unique Nash Equilibrium.

5. Come up with another example of the tragedy of the commons. Now consider the Prisoner’s Dilemma. Is this part of the tragedy of the commons family? What about the Stag Hunt?

6. Imagine a population of \( N \) stag hunters. Find the conditions on the distribution of the population that make the Nash equilibrium hunting stag the optimal strategy, and the conditions on the distribution of the population that make the Nash equilibrium hunting hare the optimal strategy.
Chapter 4

Mixed Strategy Nash Equilibria and Two Player Zero-Sum Games

We now move on to a class of games in normal form where the game, or subgame after eliminating dominated strategies, has more than one strategy available to the players and those strategies are not dominated. Consider the payoff matrix given in Figure 4.1.

We can’t use iterated elimination of dominated strategies here, since there are no dominated strategies to eliminate, and so we must come up with another solution. As before, some assumptions and definitions will help us start this process:

- Consider a GNF with players 1,...,n, with strategy sets $S_1, ..., S_n$, finite. Suppose each player has $k$ strategies, denoted $s_1, s_2, ..., s_k$.

- We say player $i$ has a mixed strategy if $i$ uses $s_i$ with a probability $p_i$. We refer to each $s_i \in S_i$ as a pure strategy.

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Move A</th>
<th>Move B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Move C</td>
<td>(4,6)</td>
<td>(7,3)</td>
</tr>
<tr>
<td>Move D</td>
<td>(8,2)</td>
<td>(1,8)</td>
</tr>
</tbody>
</table>

Figure 4.1: A GNF in which no strategy dominates another.
If the $p_i$ associated with $s_i$ is greater than zero, then we call $s_i$ an active pure strategy.

The mixed strategy of player $i$, denoted $\sigma_i$, is given by $\sigma_i = \sum_{i=1}^{k} p_i s_i$.

The following two points about playing a mixed strategy by either player will help us formulate what to do:

- If either player chooses a mixed strategy, it is because they are indifferent, or perhaps unsure, about which strategy they want to use. If they preferred one strategy over another (Move A over Move B, or Move C over Move D, for example), then they would choose that strategy over another, rather than playing randomly.

- If we assume that the mixed strategy being played is part of the Nash equilibrium of the system, then the expected payoff to a given player from choosing one move over another must be equal.

Let’s apply these ideas to the game given in Figure 4.1. For Player 1, who has the two strategies Move A and Move B, we can assign probabilities to those moves and generate an expected payoff. Let $p_A$ be the probability that Player 1 uses Move A. This implies that Player 1 will use Move B with probability $1 - p_A$. That means the expected payoff to Player 2, given that Player 1 is indifferent about his moves, using Move C is

$$6p_A + (1 - p_A)3$$

And if Player 2 uses Move D, then the expected payoff to Player 2 is

$$2p_A + (1 - p_A)8$$

Since Player 2 knows that Player 1 is indifferent about what to do, the expected payoffs to Player 2 resulting from either of her moves is equal, and we can set the expected payoffs equal to one another and solve for $p_A$.

$$6p_A + (1 - p_A)3 = 2p_A + (1 - p_A)8$$

which gives $p_A = \frac{2}{3}$. If we use similar reasoning about how Player 1 knows that Player 2 is indifferent, then we can set up a similar equality of expected payoffs, given that Player 2 will use Move C with probability $p_C$ and Move D with probability $p_D = 1 - p_C$. The resulting equations are
\[ 4p_C + (1 - p_C)8 = 7p_C + (1 - p_C)1 \]

solving this for \( p_C = 0.7 \). If both players are rational, then these probability values represent the best response of both players, and the solution is a Nash equilibrium. We can represent the mixed strategies as \( \sigma_1 = \frac{2}{3}A + \frac{1}{3}B \) and \( \sigma_2 = 0.7C + 0.3D \), and we often will write a mixed strategy profile as \( (\sigma_1, \sigma_2) \). In this case our mixed strategy profile is \( (\frac{2}{3}A + \frac{1}{3}B, 0.7C + 0.3D) \), which also happens to be our mixed strategy Nash equilibrium.

### 4.0.1 Two Theorems

The following two theorems are presented without proof, but their utility in analyzing mathematical games cannot be overstated.

**Theorem 3 Nash’s Existence Theorem**

If, in an n-person game in normal form, each player’s strategy set is finite, then the game has at least one mixed strategy Nash equilibrium.

**Theorem 4 The Fundamental Theorem of Nash Equilibria**

The mixed strategy profile \( (\sigma_1, \sigma_2, ..., \sigma_n) \) is a mixed strategy Nash equilibrium if and only if

1. Given two active strategies of player \( i \), \( s_i \) and \( s_j \), the expected payoff to player \( i \) is the same regardless of whatever mixed strategies the other players employ.

2. The pure strategy \( s_i \) is active in the mixed strategy \( \sigma_i \) for player \( i \), and the pure strategy \( s_j \) is not active in the mixed strategy \( \sigma_i \), then \( \pi_i(s_i) \geq \pi_i(s_j) \), the payoff to \( i \) when she uses \( s_i \) is greater than or equal to her payoff when she uses \( s_j \), regardless of whatever mixed strategy the other players employ.

What to take away from these theorems? Each player’s active strategies are all best responses to the profile of the other player’s mixed strategies, where “best response” means best response among pure strategies.
4.1 Two Player Zero Sum Games

A zero sum game is a game, usually in normal form but not always, that forces the total sum of the payoffs to be zero. What one player wins, the other has to lose. We will restrict our attention to the finite game. Two player zero sum games (TPZSG), also called matrix games have some interesting properties, but perhaps the most significant is the Minimax Theorem:

**Theorem 5** The Minimax Theorem
For every finite two-player zero sum game,

- There is some value $V$, called the game value.
- There exists a mixed strategy for Player 1 that guarantees an average payoff of at least $V$ to Player 1, regardless of Player 2’s strategy.
- There exists a mixed strategy for Player 2 that guarantees an average payoff of at most $-V$ (a loss) to Player 2, regardless of Player 1’s strategy.
- If both players use the strategies that guarantee their average payoffs, the mixed strategy profile is a Nash equilibrium.

This theorem is offered without proof, but it shouldn’t be too hard to see why it is true. If $V$ is zero, we define the game as fair. If $V$ is not equal to zero, we define the game as unfair. If $V$ is positive, the game is weighted in Player 1’s favour, and if $V$ is negative, the game is weighted in Player 2’s favour.

4.1.1 The Game of Odds and Evens

Players 1 and 2 simultaneously call out the numbers 1 or 2. If the sum is odd, Player 1 wins the sum of the numbers called out. If the sum is even, Player 2 wins the sum of the numbers called out. The loser must pay the winner the sum. The payoff matrix for this game is given in Figure 4.2.

Which player does this game favour? It may not be obvious from the start, but we can explicitly calculate the mixed strategy Nash equilibrium for this game, and show it is also the Minimax strategy for both players. This will also tell us which player the game favours.
4.2. DOMINATION OF TWO PLAYER ZERO SUM GAMES

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Player 1</td>
<td>(-2, 2)</td>
<td>(3, -3)</td>
</tr>
<tr>
<td>Player 2</td>
<td>(3, -3)</td>
<td>(-4, 4)</td>
</tr>
</tbody>
</table>

Figure 4.2: The Odds and Evens Game Payoff Matrix.

- Let $p$ be the probability that Player 1 uses the strategy 1, and $1 - p$ for 2.
- Let $q$ be the probability that Player 2 uses the strategy 1, and $1 - q$ for 2.

If we use the method defined earlier, multiplying probabilities and payoffs, setting equations equal to one another and solving, we end up with $p = \frac{7}{12}$ and $q = \frac{7}{12}$.

If we calculate the average payoffs to Player 1, we see that $\pi_1 = \left(\frac{7}{12}\right)^2(-2) + 2\left(\frac{7}{12}\right)\left(\frac{5}{12}\right)(3) + \left(\frac{5}{12}\right)^2(-4) = \frac{1}{12}$, and $\pi_2 = \left(\frac{7}{12}\right)^2(2) + 2\left(\frac{7}{12}\right)\left(\frac{5}{12}\right)(-3) + \left(\frac{5}{12}\right)^2(4) = \frac{11}{12}$. These payoff values are what we defined as the game value $V$. These average payoffs indicate that this game played at the Minimax strategy favours Player 1, and by the Fundamental Theorem of Nash Equilibria, this Minimax strategy is also the NE.

4.2 Domination of Two Player Zero Sum Games

When representing the payoffs to the players in a matrix game, we will now use the convention that only Player 1’s payoffs will appear in the matrix. This is due to the fact that the payoffs are symmetric, and if we wanted to display Player 2’s payoff we would just multiply every value in the matrix by -1. The Odds and Evens game would be represented as

<table>
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<tbody>
<tr>
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<td>3</td>
</tr>
<tr>
<td>Player 2</td>
<td>-2</td>
<td>-4</td>
</tr>
</tbody>
</table>
4.2.1 Saddle Points

A saddle point, reminiscent of the saddle points in multidimensional calculus, of a matrix game is the term in the matrix that is the minimum of its row and the maximum of its column. An example is given in Figure 4.3.

The value at \( a_{21} \) is 2, which is the minimum of its row and the maximum of its column. Recall that the payoffs are with respect to Player 1.

4.2.2 Solving Two by Two Games

Consider the following two by two matrix

\[
B = \begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\]

Strategy for finding the solution

- Check for a saddle point. If there is a saddle point, that value is the game value \( V \).
- If there is no saddle point, we can find the mixed strategy Nash equilibrium by finding the minimax strategy.

If there is no saddle point, we can characterize the solutions using the following equation.

\[
\begin{align*}
pa + (1 - p)c &= pb + (1 - p)d \\
(a - b)p &= (1 - p)(d - c) \\
p &= \frac{d - c}{(a - b) + (d - c)}
\end{align*}
\]

The average return to Player 1 based on this probability is

\[
V = \pi_1 = pa + (1 - p)b = \frac{(ad - bc)}{a - b + d - c}
\]
4.3. **GOOFSPIEL**

We can use a similar technique to show that $V$ is the same for Player 2 and that

$$q = \frac{d - b}{a - b + d - c}$$

**Note:** Often in TPZSG, it is necessary to reduce the game to a smaller matrix. If it happens to reduce down to a two by two matrix, then the solution to the larger game can be characterized by the solution given.

For example, if we consider the matrix game given in

$$G = \begin{pmatrix} 1 & 4 & 10 \\ 2 & 5 & 1 \\ 3 & 6 & 9 \end{pmatrix}$$

Column 2 is dominated by Column 1, and Row 1 is dominated by Row 3, so this game is now reduced to

$$H = \begin{pmatrix} 1 & 10 \\ 3 & 9 \end{pmatrix}$$

The new matrix $H$ has a saddle point, at $a_{21}$, with a value of 3. If you attempted to solve this matrix using the characterization given, you would get a probability that exceeded 1 or was less than zero.

### 4.3 Goofspiel

Goofspiel [?] is a two player zero sum game that defies complete conventional analysis. It is played with a standard 52 card deck and the rules are as follows:

- Remove 1 suit (13 cards) from the deck. This is usually clubs, but it is irrelevant which suit you remove.

- Player 1 receives the 13 cards of the hearts suit, and Player 2 receives the 13 cards of the diamonds suit.

- The 13 remaining spades are shuffled and placed in between the two players.

- One is turned face up. The two players simultaneously choose a card and discard it face up. Whichever player discarded the card with the highest value (Ace being worth 1, King is worth 13) wins the spade card that is turned up in the middle.
• The worth of the spade card is added to the player’s score, and subtracted from the other player’s score (it is possible to have negative score).

• If both players discard a card of equal value (not necessarily to the spade), then they receive nothing and the spade card is lost.

• Repeat until no cards are left. The losing player must pay their score to the winner (perhaps in dollars).

Interestingly, in work done by Sheldon in 1971, it is shown that the best course of play against an opponent that plays randomly is to match your bid to the value of the spade card. That is the only result currently known about how to play this variation of the game. Another variation is when the card in the middle is hidden, and then it is best to just play randomly.

Goofspiel has recently been featured in a round robin tournament that further shows that there is a great deal of analysis still required to get a handle on this game, if it is at all possible. At this point, we still don’t really know how to play Goofspiel.

4.4 Exercises

1. Reduce the following game $G$ and find the MSNE of the subgame $H$. Find the expected payoff to both players.

$$G = \begin{pmatrix} (4,1) & (5,3) & (6,4) \\ (8,1) & (9,5) & (1,2) \\ (2,0) & (4,1) & (0,2) \end{pmatrix}$$

2. If $G$ represents a TPZSG, find its game value if

$$G = \begin{pmatrix} 2 & 5 & 8 \\ 1 & 4 & 1 \\ 3 & 4 & 5 \end{pmatrix}$$

3. (Short Essay) Is the two player zero sum game model appropriate for war between nations? Be sure to think about what is involved in the fighting and the resolution of a war. World War 1 and World War 2
are fine examples, but there have been wars of all sizes throughout the course of recorded human history. Do some research on a few wars and give some examples about why a two player zero sum game model is or is not a good model.

4. Imagine you are playing Goofspiel and you know ahead of time that your opponent is using the strategy: *Match the value of the upturned card.* Find the strategy that maximizes your score every round, and prove it is a maximum score. Now generalize this strategy for any finitely sized set of Goofspiel, and show that your strategy still maximizes your score every round.

5. Come up with a strategy for an agent to play Goofspiel in a tournament, where you are uncertain of what other strategies will be submitted. You can write your strategy in English, or submit a finite state machine.