**Minimal Polynomials**

**Definition:** Let $K \subset F$ be a subfield of a field $F$ with $[F : K] < \infty$. The *minimal polynomial* of $\alpha \in F$ (with respect to $K$) is a monic polynomial $M_\alpha(X) = M_{\alpha,K}(X) \in K[X]$ of least degree such that $M_\alpha(\alpha) = 0$.

**Proposition 5:** Let $K \subset F$ with $[F : K] < \infty$, and let $\alpha \in F$. Then we have:

(a) If $f \in K[X]$, then $f(\alpha) = 0 \Leftrightarrow M_\alpha | f$.

(b) $M_{\alpha,K}(X)$ is irreducible over $K$.

(c) $M_{\alpha,K}$ is uniquely determined by $\alpha$ (and by $K$).

**Corollary:** The evaluation map $e_\alpha(f) = f(\alpha)$ induces a field injection

$$\overline{e}_\alpha : K[X]/(M_\alpha) \hookrightarrow F.$$ 

Thus, if $\deg(M_\alpha) = [F : K]$, then

$$K[X]/(M_\alpha) \cong F.$$ 

**Remark:** For any field $K$ it is true that

$$\deg(M_{\alpha,K}) | [F : K],$$ 

but we will prove this only in the case when $K$ is a finite field.