INSTRUCTIONS: The exam has six questions labeled 1 through 6. The exam is worth 100 points in total.

The exam is two hours in length.

To receive full credit you must explain your answers.

Write all answers on the exam. You may use the backs of pages if necessary.

Pre-approved calculator (Casio 991 or equivalent) is permitted.

Name: ____________________________________________

Student Number: ________________________________

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1. [5 pts each=15 pts]

1. Let \( n \in \mathbb{N} \). The binary repetition code of length \( n \) has the codewords 00 \cdots 0 and 11 \cdots 1. Show that it is a \([n, 1, n]\) linear code.

2. Let \( C \) be the binary code with all words of length 4 over \( F = \mathbb{F}_2 = \{0, 1\} \) as its codewords. Let \( C' \) be the single parity check code obtained by adding an extra bit at the end of each codeword in \( C \) to make the total number of 1s even. Show that \( C' \) is a linear \([5, 4, 2]\)-code.

3. Let \( C \) be the binary code of length 5 with codewords 00000, 01101, 10110, 11011. Show that \( C \) is 1-error correcting and 2-error detecting.
2. [5 pts each= 15 pts] The ISBN-10 coding system is a code $C$ over $F = \mathbb{F}_{11} = \{0, 1, 2, \ldots, 9, 10\}$. An ISBN-10 codeword $x = x_1x_2\cdots x_{10}$ is ten bit in length, and satisfies $\sum_{i=1}^{10} ix_i \equiv 0 \pmod{11}$. Suppose that $C$ has a parity check matrix

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{bmatrix}$$

1. Show that $C$ is a linear $[10, 8]$ code over $F = \mathbb{F}_{11}$.

2. Show that $C$ is a 1-error correcting code.

3. What is the missing bit to make $0257x41326$ an ISBN code?
3. [10 pts] A codeword of the code $C = \{01010, 10101\}$ is transmitted through a BSC with crossover probability $p < 1/2$, and MDD decoder $D$ is applied to the received word. Give a formula for the decoding error probability $P_{err}$ of $D$. (No need to do the actual calculation for $P_{err}$.)
4. [5pts=20pts] If $B$ is a matrix, $B^T$ stands for its transpose, and $I_r$ denotes the $r \times r$ identity matrix. Let $C$ be a linear $[n, k, d]$ code over $F = \mathbb{F}_q$.

1. What is a generator matrix of $C$?

2. What is a parity check matrix of $C$?

3. Show that $C$ is self-dual if and only if $k = n/2$. (A linear code $C$ is said to be self-dual if $C^\perp = C$.)

4. If $C$ has a generator matrix $G$ with standard form $[I_k \ A]$ where $A$ is a $k \times (n - k)$ matrix, show that a parity check matrix $H$ has standard form $[-A^T \ I_{n-k}]$. 
5. [5 pts each=25 pts] Let $C$ be a binary linear $[6, 3, 3]$ code with parity check matrix

$$H = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$ 

- What is the syndrome of a word?
- Show that two words have the same syndrome if and only if they lie in the same coset of $C$.
- Find coset leaders.
- Find syndromes.
- Decode $y = 11001$. 
6. [5pts each=15 pts]

1. Let $H$ be a parity check matrix for a linear $[n, k]$ code $C$. Prove that the minimum distance of $C$ is $d$ if and only if any set of $d-1$ columns of $H$ are linearly independent but some set of $d$ columns are linearly dependent.

2. Find the minimum weight of the linear code over $F = \mathbb{F}_3$ with parity check matrix

$$H = \begin{bmatrix} 1 & 0 & 0 & 2 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{bmatrix}.$$ 

3. Let $C$ be a $[n, k, d]$ linear code over $F = \mathbb{F}_q$. Prove the Singleton bound:

$$d \leq n - k + 1.$$