The Finite Fourier Transform

Let $N$ be a positive integer $[N] = \{1, 2, \ldots, N+1, N\}$ $= \{1, 2, 3, \ldots, N-1, 0\}$. Let $\omega = \exp \left( \frac{2\pi i}{N} \right)$, then $\omega^N = 1$ and $\omega + \omega^2 + \omega^3 + \ldots + \omega^N = 0$.

Given $f: [N] \rightarrow \mathbb{C}$ let $\hat{f}(k) = \sum_{\ell=0}^{N-1} f(\ell) \omega^{-\ell k}$ where we have set $f(0) = f(N)$; in fact we set $f: \mathbb{Z} \rightarrow \mathbb{C}$ by extending $f$ by periodicity. We can also think of $f$ as a vector $\hat{f} = \begin{pmatrix} f(1) \\ f(2) \\ \vdots \\ f(N) \end{pmatrix} \in \mathbb{C}^N$. Then we can write $\hat{f}$ in terms of $f$ and a matrix $W$

$$W = (\omega^{\ell i})_{\ell,i=1}^{N} \quad \text{and} \quad \omega^{ij} = \omega^{-ij} \cdot \frac{1}{\sqrt{N}} W \omega$$

the Finite Fourier Transform matrix. $\hat{f}$ is the
\( \frac{1}{\sqrt{N}} W \) is a unitary matrix: 
\[
(WW^*)_{ij} = \sum_{k=1}^{N} \overline{w_{ik}} w_{jk} = \sum_{k=1}^{N} w^{-i} k w^* j k = \sum_{k=1}^{N} w^{(i-j)k} 
\]

If \( i = j \) then \( \sum_{k=1}^{N} w^{(i-j)k} = N \). If \( i \neq j \) then \( \sum_{k=1}^{N} w^{(i-j)k} = 0 \) because \( v = w^{(i-j)} \) is a root of unity with an order which divides \( N \) and \( v \neq 1 \). Thus \( WW^* = N I_N \) and \( W^* W = N I_N \), so \( \frac{1}{\sqrt{N}} W \) is a unitary matrix. This means we can recover \( f \) from \( \hat{f} \); \( \hat{f} = Wf \) means
\[
f = \frac{1}{N} W^* \hat{f} \left( \frac{1}{N} \sum_{k=0}^{N-1} w^k \hat{f}(k) = f(i) \right) \]

**Inversion Formula**

**Convolution** Suppose \( f, g \in C^N \) which we regard as functions on \( \mathbb{Z} / N \mathbb{Z} = \{0, 1, …, N-1\} \) or as periodic functions on \( \mathbb{R} \) with period \( N \). We let
\[
f * g(k) = \sum_{\ell=0}^{N-1} f(\ell) g(k-\ell)
\]
using our convention that \( g(-\ell) = g(N-\ell) \) for \( -N \leq \ell \leq 0 \).
We think of \( C[\mathbb{N}] \) as functions \( f: \mathbb{N} \rightarrow \mathbb{C} \).

Given \( f, g \in C[\mathbb{N}] \), we let \( (af)(k) = af(k) \) and \( (f+g)(k) = f(k) + g(k) \). This makes \( C[\mathbb{N}] \) a vector space; in fact this is the same as thinking of \( f = \left( \begin{array}{c} f(0) \\ \vdots \\ f(N) \end{array} \right) \in \mathbb{C}^N \).

We can give \( C[\mathbb{N}] \) two products:

- Pointwise product: \( (f \cdot g)(k) = f(k)g(k) \)
- Convolution product: \( f \ast g(k) = \sum_{l=0}^{N-1} f(l)g(k-l) \)

So we can make \( C[\mathbb{N}] \) into an algebra two different ways. Note that the convolution product is associative and commutative:

\[ f \ast g(k) = \sum_{i,j=0}^{N-1} f(i)g(j) \quad \text{if} \quad i+j \equiv k \pmod{N} \]

The pairs are \( (1, k-1), (2, k-2), \ldots, (k-1, 1), (k, N), \ldots \)

So
\[ f \ast g(k) = \sum_{j=1}^{N} f(k-j)g(j) \]

Now the convolution product is commutative and associative.
\[ f \ast g (k) = \sum_{i,j} f(i) g(j) = g \ast f (k) \]

Where \( (f \ast g) \ast h = f \ast (g \ast h) \). Now we have two algebras \( (C([0,N]), \cdot) \) and \( (C([0,N]), *) \), but we see what happens when we use the Fourier transform:

\[
\hat{f} \ast \hat{g}(k) = \hat{f}(\omega) \hat{g}(k) = \sum_{\omega = 0}^{N-1} \sum_{l=0}^{N-1} f(l) e^{-i \omega k} g(l) e^{-i \omega \bar{k}}
\]

\[
= \sum_{l=0}^{N-1} f(l) \sum_{\omega = 0}^{N-1} g(l) e^{-i \omega \bar{k}} e^{i \omega k} \]

\[
= \sum_{l=0}^{N-1} f(l) \left[ \sum_{\omega = 0}^{N-1} g(l) e^{-i (\omega - \bar{k}) k} \right] e^{-i \omega \bar{k}}
\]

\[
= \sum_{\bar{k}} f \ast g (\bar{k}) e^{-i \omega \bar{k}} = \hat{f \ast g}(k).
\]

Thus \( f \mapsto \hat{f} : (C([0,N]), *) \rightarrow (C([0,N]), \cdot) \) is an algebra homomorphism, which is one-to-one and onto.
Group Characters

Let $G$ be a finite abelian group. A function $f : G \rightarrow \mathbb{T}$ is a character if $f(1) = 1$ and $f(gh) = f(g)f(h)$ for all $g, h \in G$. If $G = \mathbb{Z}/N = \{0, 1, 2, \ldots, N-1\}$ with addition modulo $N$, then a character is determined by $f(1)$ for $f(k) = f(1)^{k/N}$. Thus for each character $f$ there is $\sqrt[N]{f(1)}$ such that $f(k) = \sqrt[N]{f(1)}^k$. We must have $f(0) = 1$ and so $\sqrt[N]{f(0)} = f(0) = 1$. Thus $\sqrt[N]{f(1)}$ is some $N$th root of unity, and any root of unity will do. Thus for each character $f$ there is $j = 0, 1, \ldots, N-1$ such that $f(k) = \omega^k$, let $j = 0, 1, \ldots, N-1$ and let $f$ be the character $f(k) = \omega^k$. Find $\hat{f}$, the Fourier transform of $f$. $\hat{f} = \left( \begin{array}{c} \omega^0 \\ \omega^1 \\ \vdots \\ \omega^{N-1} \end{array} \right)$ so $\hat{f}$ has a "spike" at $j$. 
6. Recall: 

\[ f(k) = \frac{1}{N} \sum_{\ell=0}^{N-1} f(\ell) \cos\left(\frac{2\pi \ell k}{N}\right) + \frac{i}{N} \sum_{\ell=0}^{N-1} f(\ell) \sin\left(\frac{2\pi \ell k}{N}\right) \]

when \( N > 2M \)

\[
\hat{f}(k) = a_0 + \sum_{j=1}^{M} a_j \cos\left(\frac{2\pi j k}{N}\right) + b_j \sin\left(\frac{2\pi j k}{N}\right)
\]

\[
a_0 = \frac{\hat{f}(0)}{N} \quad a_M = \frac{\hat{f}(M)}{N} \quad b_M = 0
\]

\[
a_j = \frac{\hat{f}(0) + \hat{f}(N-j)}{N} \quad b_j = \frac{\hat{f}(j) - \hat{f}(N-j)}{N}
\]

\(1 \leq j \leq M-1\)

Filters \[ \hat{g} = \begin{array}{cccccc}
1 & 2 & 3 & \cdots & 1
\end{array} \]

\[ \hat{f} \hat{g} \text{ turns off certain frequencies} \]

\[ \hat{f} \hat{g} = \hat{f} \ast \hat{g} \text{ i.e. convolving with } \hat{g} \]

\[ \hat{f} \hat{g} \text{ turns off unwanted frequencies} \]

**Fourier Analysis on \( \mathbb{T} \)**

\[ \mathbb{T} = \mathbb{R}/\mathbb{Z} \text{ but we shall use the picture } \mathbb{T}' = \mathbb{R}/(2\pi \mathbb{Z}) \]
\[ [\pi, \pi] \ni \theta \rightarrow \mathrm{e}^{i\pi} \in \mathrm{I} \ni \{z \mid |z| = 1\} \]

\[ \mathrm{e}^{i\pi} = \mathrm{e}^{-i\pi} = -1 \]

A periodic function \( f \) on \( \mathbb{R} \)
periodic means \( f(t + 2\pi) = f(t) \)

\[ a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \, dt, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) \, dt \quad n \geq 1 \]

\[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) \, dt \]

\( \{a_n\}_{n=0}^\infty \) = Fourier cosine coefficients of \( f \)

\( \{b_n\}_{n=1}^\infty \) = Fourier sine coefficients of \( f \)

\[ \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos(nt) + b_n \sin(nt) \right] = \text{Fourier series of } f. \]
Example: a square wave

\[ a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \, dt = 0 \]

\[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt \, dt = \frac{1}{\pi} \int_{-\pi}^{0} \cos nt \, dt + \frac{1}{\pi} \int_{0}^{\pi} \cos nt \, dt \]

\[ = \frac{1}{\pi} \left\{ \left[ \frac{-\sin nt}{n} \right]_{-\pi}^{0} + \left[ \frac{\sin nt}{n} \right]_{0}^{\pi} \right\} = 0 \]

\[ b_n = \frac{1}{\pi} \left\{ \int_{-\pi}^{\pi} f(t) \sin nt \, dt \right\} = \frac{1}{\pi} \left\{ \int_{-\pi}^{0} \sin nt \, dt + \int_{0}^{\pi} \sin nt \, dt \right\} \]

\[ = \frac{1}{\pi} \left\{ \left[ \frac{-\cos nt}{n} \right]_{-\pi}^{0} + \left[ \frac{-\cos nt}{n} \right]_{0}^{\pi} \right\} = \frac{1}{\pi} \left\{ \frac{1 - (-1)^n}{n} + \frac{(-1)^n - 1}{n} \right\} \]

\[ = \frac{2}{n\pi} (2 - 2(-1)^n) = 0 \text{ if } n \text{ is odd} \]

\[ = \frac{4}{n\pi} \text{ if } n \text{ is even} \]

Fourier series: \( b_0, \sin t + b_2 \sin 3t + b_4 \sin 5t + \ldots \)

\[ = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin((2n-1)t)}{2n-1} \]

Does this converge to what kind of convergence?

The Functions of Fourier Series

\( C([-\pi, \pi]) = \{ f : [-\pi, \pi] \rightarrow \mathbb{C} | f \text{ is continuous} \} \)