uniform $\forall \epsilon > 0 \exists N. s.t. \forall n > N, \| f_n - f \| < \epsilon$.

Uniqueness of Fourier Series

Question: given $f, g$ integrable with $\hat{f}(n) = \hat{g}(n) \forall n$ does $f = g$?

Theorem: If $f$ is integrable and $f(n) = 0 \forall n$ then $f(\theta) = 0$ for every $\theta$ at which $f$ is continuous.

\[
\hat{f}(n) = \hat{f}(-n). \text{ If } f = \text{even} \text{ then } \hat{u}(n) = \frac{\hat{f}(n) + \hat{f}(-n)}{2} = 0, \hat{v}(n) = \frac{\hat{f}(n) - \hat{f}(-n)}{2} = 0
\]

so we may suppose $f$ is real valued

Proof: Suppose $f$ is continuous at $\theta_0$. We shall show that $f(\theta_0) = 0$. Let $g(\theta) = f(\theta + \theta_0)$, let us compute

\[
\hat{g}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\theta) e^{-in\theta} d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta + \theta_0) e^{-in\theta} d\theta = \frac{1}{2\pi} \int_{\theta_0}^{\theta + \theta_0} f(\theta) e^{-in(\theta - \theta_0)} d\theta = e^{in\theta_0} \hat{f}(n) \quad \{ \theta = \theta + \theta_0 \}
\]
\( g(n) = e^{i\pi n} f(n) \). So if \( f(n) = 0 \) for all \( n \) then \( g(n) = 0 \) for all \( n \). Moreover, \( g(0) = f(\theta) \) and if \( f \) is continuous at \( \theta \) then \( g \) is continuous at \( 0 \). Thus it suffices to show \( g(0) = 0 \).

So we suppose \( g(0) \neq 0 \), and reach a contradiction. We may suppose \( g(0) > 0 \), otherwise we replace \( g \) by \(-g\). We shall call a function of the form

\[
p(\theta) = \sum_{n=-N}^{N} a_n e^{i\pi n \theta}
\]

a trigonometric polynomial. Since \( g(n) = 0 \) for all \( n \), we have that

\[
\int_{-\pi}^{\pi} g(\theta) p(\theta) d\theta = \sum_{n=-N}^{N} a_n \int_{-\pi}^{\pi} g(\theta) e^{i\pi n \theta} d\theta = 0
\]

for every trigonometric polynomial \( p \). If we could find \( \varepsilon > 0 \), \( \delta > \eta > 0 \), such that for each \( k \), there is a trigonometric polynomial \( p \) such that

- \( |p_k(\theta)| > (1+\varepsilon)^k \) for \( 1 \theta < \eta \)
- \( |p_k(\theta)| < (1-\varepsilon)^k \) for \( \delta < 1 \theta < \pi \)
- \( p_k(\theta) > 0 \) for \( \eta < 1 \theta < \pi \)
then we could reach our contradiction.

\[ 0 = \int_{-\pi}^{\pi} g(\theta) \, p_k(\theta) \, d\theta \]

\[ = \int_{\theta_1 \leq \theta \leq \theta_2} g(\theta) \, p_k(\theta) \, d\theta + \int_{\theta_3 \leq \theta < \theta_4} g(\theta) \, p_k(\theta) \, d\theta \]

\[ + \int_{\theta_5 \leq \theta < \pi} g(\theta) \, p_k(\theta) \, d\theta = I_1 + I_2 + I_3 \]

and we can show \( I_1 \to 0, I_2 \geq 0, I_3 \to 0 \).

As then for \( k \) large enough \( I_1 \geq 1, |I_3| < 1 \) and

\[ 0 = I_1 + I_2 + I_3 > I_1 + I_2 - 1 > I_1 - 1 \geq 1. \]

Step I. Find \( \varepsilon_1, n, p_k \) as above.

Choose \( \varepsilon > 0 \) so that

\[ \varepsilon < \frac{1 - \cos \delta}{\delta}; \text{ then} \]

for \( \delta \leq 0 \leq \pi \) we have

\[ \cos \theta < \cos \delta < 1 - \varepsilon \] so \( 2\varepsilon + \cos \theta < 1 - \varepsilon. \)

Also \( -1 < \cos \theta \) so \( -1 + \varepsilon < 2\varepsilon + \cos \theta \) for all \( \theta \).

Hence for \( \delta \leq 0 \leq \pi \) we have \( 2\varepsilon + \cos \theta < 1 - \varepsilon. \)
Next choose $0 < \eta < \delta$ so that for
$101 < \eta$ we have $\cos \theta > 1 - \varepsilon$. Then
for $101 < \eta$, $2\varepsilon + \cos \theta > 1 + \varepsilon$.

Now let $p_k(\theta) = (2\varepsilon + \cos \theta)^k$. By
our discussion with Chebyshev polynomials
we have $p_k$ is a trigonometric polynomial
and satisfies the three conditions above.

Now $I_1 = \int g(\theta) p_k(\theta) d\theta \geq g(\theta) \int_{101}^{\eta} p_k(\theta) d\theta$

$\geq \frac{g(\eta)}{2} (1 + \varepsilon)^k \cdot 2\eta$.

$I_2 = \int_{\eta}^{101} g(\theta) p_k(\theta) d\theta \geq \frac{g(101)}{2} \int_{\eta}^{101} p_k(\theta) d\theta$

$\geq 0$

$|I_3| = \int_{\delta}^{\eta} |g(\theta) p_k(\theta)| d\theta \leq M \int_{\delta}^{\eta} |p_k(\theta)| d\theta$

$\leq M 2(\pi - \delta)(1 + \varepsilon)^k$

where $M = \sup_{101 < \eta} |g(\theta)|$ which is finite
because we assumed that $f \in$ Riemann
integrable and thus bounded.
Corollary: If $f$ is continuous on $[-\pi, \pi]$ and $f(-\pi) = f(\pi)$ and $\hat{f}(n) = 0 \forall n$ then $f = 0$.

Corollary

Suppose $f$ is continuous on $T$ (i.e. $f$ is continuous on $[-\pi, \pi]$ and $f(\pi) = f(-\pi)$) and $\sum |\hat{f}(n)| < \infty$. Then $\sum \frac{S_n(f)}{n} \rightarrow$ converges uniformly to $f$ on $T$.

Proof: Note first that $\sum |\hat{f}(n)| = \sum |\hat{f}(n)|$, so by the Weierstrass M-test the series $\sum \frac{S_n(f)}{n}$ converges uniformly to a function $g$ on $T$. Since the convergence is uniform and $\theta \mapsto \hat{f}(n)e^{in\theta}$ is continuous, $g$ is continuous. Let us compute $g(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta$.

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \lim_{N \to \infty} S_n(f)(\theta) e^{-in\theta} d\theta$$

$$= \lim_{N \to \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} S_n(f)(\theta) e^{-in\theta} d\theta$$

(by uniform convergence)

$$= \lim_{N \to \infty} \sum_{n=-N}^{N} \hat{f}(m) \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{im-n\theta} d\theta = \lim_{N \to \infty} \sum_{m=-N}^{N} \hat{f}(m) \delta_{m,n}$$
Thus $g$ is a continuous function on $\mathbb{T}$ with $g(n) = \hat{f}(n) \forall n$. Hence $f-g$ is a continuous function on $\mathbb{T}$ with $\hat{f-g}(n) = 0$, Hence by the previous corollary $f-g = 0 \implies f=g$. 

This means that $\{S_N(f)\}_{N=1}^{\infty}$ converges uniformly to $f$ as claimed.

**Corollary**

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

**Proof:** Let $f(\theta) = \frac{(\pi - \theta)^2}{4}$ on $[0, 2\pi]$. $f(0) = \frac{\pi^2}{4} = f(2\pi)$. So the periodic extension of $f$ to $\mathbb{R}$ gives a continuous function on $[-\pi, \pi]$ with $f(\pi) = f(-\pi)$. Note that for this extension $f(\theta) = \frac{(\pi + \theta)^2}{4}$ when $-2\pi \leq \theta \leq 0$. Now we computed earlier $\hat{f}(n) = \frac{1}{2n^2}$ for $n \neq 0$ and $\hat{f}(0) = \frac{\pi^2}{12}$.

Thus for $0 \leq \theta \leq \pi$ we have

$$\frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{\sin(\theta) e^{\imath n \theta} + e^{\imath n \theta}}{2n^2} = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{\cos(n\theta)}{n^2} = \frac{(\pi - \theta)^2}{4}$$
At $\Theta = 0$ we have
\[ \frac{x^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \]
solving we get
\[ \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \]

Notation: Let $\{a_n\}$ be a sequence of complex numbers. We say $a_n = O\left(\frac{1}{n^2}\right)$ if there is $C > 0$ such that $|a_n| \leq \frac{C}{n^2}$ for $n \geq 1$.

This is called Landau's big $O$ notation.

Corollary: Suppose $f$ is a function on $\mathbb{T}$ such that $f''$ exists and is continuous on $\mathbb{T}$. Then $\hat{f}(n) = O\left(\frac{1}{n^2}\right)$ and the Fourier series of $f$ converges uniformly (and absolutely) to $f$.

Proof: We only have to prove $\hat{f}(n) = O\left(\frac{1}{n^2}\right)$ and then we may apply a previous corollary.

Important Formula:

If $f$ has a continuous derivative on $\mathbb{T}$ then $\hat{f'}(n) = i\pi \hat{f}(n)$.
For $n \neq 0$

$$2\pi f(n) = \int_{-\pi}^{\pi} f(\theta) e^{in\theta} d\theta = \int_{-\pi}^{\pi} f(\theta) \frac{e^{in\theta}}{-in} df$$

$$= \left[ \frac{f(\theta) e^{-in\theta}}{-in} \right]_{-\pi}^{\pi} + \frac{1}{in} \int_{-\pi}^{\pi} e^{in\theta} f'(\theta) d\theta$$

$$= \frac{f(\pi) e^{in\pi} - f(-\pi) e^{-in\pi}}{-in} + \frac{2\pi \hat{f'}(n)}{in} = \frac{2\pi}{in} \hat{f'}(n)$$

Hence \( \hat{f'}(n) = in \hat{f}(n) \). Finally

\[ \hat{f'}(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f'(\theta) d\theta = \frac{f(\pi) - f(-\pi)}{2\pi} = 0. \]

Hence for all \( n \)

\[ \hat{f'}(n) = in \hat{f}(n). \]

If \( f' \) has a continuous derivative on \( T \) then

\[ \hat{f''}(n) = in \hat{f'}(n) = -n^2 \hat{f}(n). \]

Now since \( f'' \) is continuous on \( T \) it is bounded; let \( M \) be such that \( |f''(\theta)| \leq M \) for all \( \theta \). Then

\[ |\hat{f''}(n)| = \frac{1}{2\pi} \left| \int_{-\pi}^{\pi} f''(\theta) e^{-in\theta} d\theta \right| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f''(\theta)| d\theta \leq \frac{M}{2\pi} \int_{-\pi}^{\pi} d\theta = M. \]
\[
\text{So } |f''(n)| \leq M. \text{ Hence } \lim_{n \to 0} \left| \hat{f}(n) \right| = \frac{1}{n^2} \leq \frac{M}{n^2} \text{ if } \\
\hat{f}(n) = O\left( \frac{1}{n^2} \right).
\]

**Convolution on } [-\pi, \pi]}

Suppose \( f \) and \( g \) are Riemann integrable and 2\( \pi \)-periodic. Then for all \( x \), the function \( y \mapsto f(y)g(x-y) \) is integrable on \([ -\pi, \pi ]\). So we let

\[
f \ast g (x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)g(x-y) \, dy.
\]

\( f \ast g \) is the convolution of \( f \) and \( g \).

In Lemma 1.2 of the appendix it is shown that if \( f \) is integrable on \([a,b]\) and \( g \) is continuous on \( \mathbb{R} \) then \( g \circ f \) is integrable.

This implies in particular that \( f^2 \) is integrable whenever \( f \) is integrable. Note that \( fg = \frac{1}{4} \int (f+g)^2 - (f-g)^2 \),

so if \( f \) and \( g \) are integrable then \( fg \) is.