Main idea
\[ f(x-y)K_n(y) \to 0 \text{ for } |y| > 5 \]
\[ f(x-y)K_n(y) \approx f(x) \text{ for } |y| < 5 \]

**Theorem**

Suppose \( f \) is integrable on \([-\pi, \pi]\) and \( \{K_n\}_n \) is a sequence of good kernels. Then
\[
\lim_{n \to \infty} f \ast K_n(\theta) = f(\theta) \text{ for each } \theta \text{ at which } f \text{ is continuous. If } f \text{ is continuous on } \mathbb{T}, \text{ then } \{f \ast K_n\}_n \text{ converges uniformly to } f \text{ on } \mathbb{T}.
\]

**Proof:** Suppose \( f \) is continuous at \( x \). Let \( M \) be as in (**) and let \( \varepsilon > 0 \) be given. Choose \( \delta > 0 \) so that \( |f(x-y)-f(x)| < \frac{\varepsilon}{M} \) for \( |y| < \delta \). Let \( B = \sup_{-\pi \leq \theta \leq \pi} |f(\theta)| \) and choose \( N \) so that
\[
\int_{S \geq |y|\delta} |K_n(x)| dx < \frac{\varepsilon}{2B}.
\]
Then for \( n \geq N \)

\[
\left| f \ast K_n (x) - f(x) \right| \leq \frac{1}{2 \pi} \int_{-\pi}^{\pi} \left| K_n(y) \right| \left| f(x-y) - f(x) \right| dy
\]

\[
= \frac{1}{2 \pi} \int_{-\pi}^{\pi} \left| K_n(y) \right| \left| f(x-y) - f(x) \right| dy
\]

\[
\leq \frac{C}{2M} \int_{-\pi/2}^{\pi/2} \left| K_n(y) \right| dy + \frac{B}{\pi} \int_{-\pi/2}^{\pi/2} \left| K_n(y) \right| dy
\]

\[
\leq \frac{C}{2} + \frac{B}{\pi} \frac{\pi}{2B} = \varepsilon.
\]

**Cesàro Summability**

*From Ramanujan's 1913 letter to Hardy*

Let 

\[
C = 1 + 2 + 3 + 4 + \ldots + 6 + 7 + 8 + 9 + 10
\]

\[
-4C = \quad 4 + \quad 3 + \quad 12 + \quad 16 + \ldots
\]

\[
-3C = \quad 1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + 9 - 10 + \ldots
\]

Now 

\[
\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \ldots \quad \text{apply } \frac{d}{dx}
\]

\[
\frac{-1}{(1+x)^2} = -1 + 2x - 3x^2 + 4x^3 - 5x^4 + \ldots
\]

\[
\frac{1}{(1+x)^2} = \quad 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \ldots
\]
\[ @ x=1 \text{ we get } \]
\[ \frac{1}{4} = 1 - 2 + 3 - 4 + 5 - 6 + \ldots = -3c \]

Thus \[ C = \frac{1}{12} \text{ i.e. } 1 + 2 + 3 + \ldots = \frac{1}{12} \]

Let \[ \sum_{n=1}^{\infty} c_n \] be a series and \[ s_n = \sum_{k=1}^{n} c_k \]
be its \( n \)th partial sum. If the sequence \( s_n \) converges then we say \[ \sum_{n=1}^{\infty} c_n \] converges to \( s \).

Now let \[ \tau_n = \frac{1}{n} (s_1 + \ldots + s_n) \]. If \( \tau_n \) converges to \( s \) we say that \[ \sum_{n=1}^{\infty} c_n \]
do Cesàro summable to \( s \).

Example

\[ c_n = (-1)^{n+1} \]
\[ s_1 = c_1 = 1 \]
\[ s_2 = c_1 + c_2 = 0 \]
\[ s_3 = c_1 + c_2 + c_3 = 1 \]

\[ s_{2n} = 0 \quad s_{2n-1} = 1 \quad s_0 \]

\[ \tau_1 = \frac{1}{1} s_1 = 1 \]
\[ \tau_2 = \frac{1}{2} (s_1 + s_2) = \frac{1}{2} \]
\[ \tau_3 = \frac{1}{3} (s_1 + s_2 + s_3) = \frac{3}{2} \]
\[ \tau_4 = \frac{1}{4} (s_1 + s_2 + s_3 + s_4) = \frac{1}{2} \]
In general \( J_{2n} = \frac{1}{2^n} (S_1 + \cdots + S_{2n}) = \frac{1}{2} \)

Thus \( \lim_{n \to \infty} J_n = 1 \). Hence

\[ 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \text{ is Cesàro summable to } \frac{1}{2}. \]

\[ S_1 = C_1 \]
\[ S_2 = C_1 + C_2 \]
\[ S_3 = C_1 + C_2 + C_3 \]
\[ \vdots \]
\[ S_n = C_1 + C_2 + C_3 + \cdots + C_{n-1} + C_n \]

\[ S_1 + \cdots + S_n \]
\[ = nC_1 + (n-1)C_2 + \cdots + (n-(n-2))C_{n-1} + (n-(n-1))C_n \]
\[ = \sum_{k=1}^{n} (n-(k-1))C_k \]

\[ J_n = \sum_{k=1}^{n} (1 - \frac{k-1}{n})C_k \]

Fejér's Kernel

Recall \( D_N(\Theta) = \sum_{n=-N}^{N} e^{i n \Theta} = \frac{\sin((N+\frac{1}{2})\Theta)}{\sin(\Theta)} \)

Let \( F_N(\Theta) = D_0(\Theta) + \cdots + D_{N-1}(\Theta) \)

Let \( D_N(f)(\Theta) = S_0(f)(\Theta) + \cdots + S_{N-1}(f)(\Theta) \)

\[ = f \star D_0(\Theta) + \cdots + f \star D_{N-1}(\Theta) \]

\[ = (f \star D_0 + \cdots + D_{N-1})(\Theta) \]
\[ f \ast F_N(\theta) = \frac{\sin^2(N\theta/2)}{\sin^2(\theta/2)} \cdot N \cdot \frac{1 - \cos N\theta}{1 - \cos \theta} \]

Lemma \( F_N(\theta) = \frac{1}{N} \cdot \frac{\sin^2(N\theta/2)}{\sin^2(\theta/2)} \cdot \frac{1 - \cos N\theta}{1 - \cos \theta} \) (Exercise)

Fejér's Theorem

The sequence \( F_N(\theta) \) is a sequence of good kernels.

Proof: Note that \( F_N(\theta) \geq 0 \) and since
\[ \int_{-\pi}^{\pi} F_N(\theta) d\theta = 1 \quad \text{for all } N \geq 0 \] we have
\[ \int_{-\pi}^{\pi} F_N(\theta) d\theta = 2\pi \int_{-\pi}^{\pi} |F_N(\theta)| d\theta = 1 \]

Let us show that \( \forall \delta > 0 \lim_{N \to \infty} \int_{|\theta| \geq \delta} |F_N(\theta)| d\theta = 0 \).

So let \( \delta > 0 \) begin. For \( \delta \) is \( \pi \)

Thus \( |F_N(\theta)| \leq \frac{1}{N} \cdot \frac{2}{1 - \cos \delta} \). So
\[ \int_{-\pi}^{\pi} |F_N(\theta)| d\theta \leq \frac{1}{N} \frac{\pi}{1 - \cos \frac{2\pi}{N}} \] 

Thus \( \sum F_N(x) \) is a sequence of good kernels.

**Theorem**

Suppose \( f \) is integrable on \([\!-\pi, \pi]\!), then \( \sum S_N(f) \) is Cesàro summable to \( f \) at every \( \theta \) at which \( f \) is continuous. If \( f \) is continuous on \( T \) then \( \sum S_N(f) \) converges uniformly to \( f \) on \( T \).

**Proof:** This follows immediately from the Theorem on p. 34.

**Corollary** If \( f \) is continuous at \( \theta \) and \( f = 0 \) then \( f(\theta) = 0 \).

**Corollary** If \( f \) is continuous on \( T \) then \( f \) can be uniformly approximated by trigonometric polynomials.

**Remark:** \( e^{in\theta} = \sum_{m=0}^{\infty} \frac{\sin m\theta}{m!} \theta^m \) is a uniform limit of polynomials.

Thus with some work we can show that \( f \) continuous