1. (10 pts) (Linearization of a nonlinear control system) Recall that if \( F : \mathbb{R} \to \mathbb{R}^p \) is a \( C^2 \) function, then, \( \forall \tau_0, \tau_1 \in \mathbb{R} \), we have the Taylor expansion of \( F \) (around \( \tau_0 \), evaluated at \( \tau_1 \)):

\[
F(\tau_1) = F(\tau_0) + (\tau_1 - \tau_0) F'(\tau_0) + R_2(\tau_0, \tau_1),
\]

where the remainder term \( R_2 \) is given by

\[
R_2(\tau_0, \tau_1) = \int_{\tau_0}^{\tau_1} (\tau_1 - t) F''(t) dt.
\]

Let now \( f : \mathbb{R}^n \to \mathbb{R}^p \) be given by

\[
f = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_p \end{pmatrix},
\]

where the \( f_i \) are \( C^2 \).

Let \( x \in \mathbb{R}^n \) be given by

\[
x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix};
\]

\( f \) therefore maps \( x \) to the vector

\[
\begin{pmatrix} f_1(x_1, \ldots, x_n) \\ f_2(x_1, \ldots, x_n) \\ \vdots \\ f_p(x_1, \ldots, x_n) \end{pmatrix}.
\]

Let now \( a, b \in \mathbb{R}^n \). Using the above Taylor expansion, show that we have:

\[
f(b) = f(a) + Df|_a \cdot (b - a) + R_2(a, b),
\]

where the remainder term \( R_2 \) is an integral with the integrand involving only the second partial derivatives of the \( f_i \) (you don’t have to compute it explicitly) and where \( Df|_a \) is the \( p \times n \) matrix with entry on row \( i \) and column \( j \) given by the partial derivative \( \frac{\partial f_i}{\partial x_j} \) of \( f_i \) with respect to \( x_j \), evaluated at \( a \). (Hint: Define \( F : \mathbb{R} \to \mathbb{R}^p \) by \( F(t) = f(a + t(b - a)) \), with \( t \in \mathbb{R} \); then Taylor expand \( F \) around \( \tau_0 = 0 \) and evaluate at \( \tau_1 = 1 \) using the chain rule.)

2. (10 pts) (Linearization of a nonlinear control system) Consider the nonlinear control system described by:

\[
\ddot{x}(t) + (\sin t) \dot{x}(t) + x^2(t) = u(t), \\
y(t) = \dot{x}^2(t),
\]

where \( t > 0 \).

(a) Show (by direct verification) that \( t \mapsto (\dot{x}(t), \ddot{u}(t), \ddot{y}(t)) = (\cos t, \cos 2t - \cos t, \sin^2 t) \) is a trajectory of this nonlinear control system.
(b) Linearize the above nonlinear control system around the trajectory given in (a) and express the result in standard LTV form.

3. (10 pts) (Banach Fixed Point Theorem) On the complete metric space \((\mathbb{R}, d)\) (with \(d(x, y) = |x - y|, \forall x, y\)), construct a mapping \(T : \mathbb{R} \to \mathbb{R}\) which satisfies \(d(T(x), T(y)) < d(x, y)\) for all \(x, y \in \mathbb{R}\) with \(x \neq y\), and such that \(\forall x \in \mathbb{R} : T(x) \neq x\) (i.e. \(T\) has no fixed point in \(\mathbb{R}\)). This shows that the condition \(d(T(x), T(y)) \leq Cd(x, y)\) (with \(0 \leq C < 1\)) in the Banach fixed point theorem cannot be relaxed.

4. (10 pts) (Possible pathological behavior of nonlinear control systems) Consider the nonlinear control system governed by the scalar ODE

\[
\dot{x}(t) = 4x^2(t) + u(t), \quad t > 0, \\
x(0) = 1,
\]

with \(x\) being the state function (aka “state trajectory”) and \(u\) the control function (both assumed real-valued). Let us now choose the control function to be identically 0, i.e. assume \(u(t) = 0\) \(\forall t \geq 0\). Show (by direct calculation) that the trajectory of this nonlinear control system, with this choice of control function, is defined only on a proper sub-interval \([0, T]\) of \([0, \infty]\) and cannot be extended to a \(C^1\) function on all of \([0, \infty]\). Find the largest such \(T\). This shows that for nonlinear control systems, the state trajectory need not be defined for all time.

5. (10 pts) (Further possible pathological behavior of nonlinear control systems) Consider the nonlinear control system governed by the scalar ODE

\[
\dot{x}(t) = 4x^{3/4}(t) + u(t), \quad t > 0, \\
x(0) = 0,
\]

declared for \(t \geq 0\). Let again the control function be identically 0, i.e. \(u(t) = 0\) \(\forall t \geq 0\). With this choice of control function, the constant zero function \(x : [0, \infty] \to \mathbb{R}\) defined by \(t \mapsto x(t) = 0\) is a solution of the above ODE with the given initial condition, i.e. is a state trajectory of that system (for that choice of control). For any \(a \geq 0\), define the \(C^1\) function \(x_a : [0, \infty] \to \mathbb{R}\) by \(x_a(t) = 0\) for \(t \leq a\) and \(x_a(t) = (t - a)^4\) for \(t > a\).

(a) Show that \(\forall a \geq 0\), \(x_a\) is a solution of the above ODE (with control function identically zero).

(b) Explain precisely why for that choice of control function, the state trajectory is not uniquely defined.