1. (10 pts) Find the control $u$ and final time $T$ such that the scalar system $\dot{x}(t) = u(t)$ ($t \geq 0$) is driven from $x(0) = 0$ to $x(T) = 1$ while minimizing (over $(u, T) \in C^0([0, T]; \mathbb{R}) \times [0, \infty]$) the cost function $\eta : (u, T) \mapsto \eta(u, T) = \int_0^T u^2(t)dt + T$.

2. (10 pts) Find the control $u \in C^0([0, 1]; \mathbb{R})$ such that the scalar system
\[ \dot{x}(t) = -x(t) + u(t) \]
is driven from $x = 1$ at $t = 0$ to $x = 0$ at $t = 1$ and the cost
\[ \eta(u) = \int_0^{1/2} u^2(t)dt + 2 \int_{1/2}^1 u^2(t)dt \]
is minimized.

3. (10 pts) Let $(A, B, C)$ be a minimal LTI realization, and consider the matrix Riccati equation:
\[ \begin{align*}
\dot{K}(t) &= -A^T K(t) - K(t)A + K(t)BB^T K(t) - C^T C, & t < 0, \\
K(0) &= 0;
\end{align*} \]
We have shown in class that this equation has a unique $C^1$ solution defined on $]-\infty, 0]$, which we denote by $t \mapsto \Pi(t, 0, 0)$, and we have shown that the limit $\Pi_\infty = \lim_{t \to -\infty} \Pi(t, 0, 0)$ does exist. Show that $\Pi_\infty$ satisfies the Algebraic Riccati Equation (ARE):
\[ A^T \Pi_\infty + \Pi_\infty A - \Pi_\infty BB^T \Pi_\infty + C^T C = 0. \]

4. (10 pts) Let $(A, B, C)$ be a minimal LTI realization, and let $t \mapsto \Pi(t, 0, 0)$ be the unique $C^1$ solution of the matrix Riccati equation:
\[ \begin{align*}
\dot{K}(t) &= -A^T K(t) - K(t)A + K(t)BB^T K(t) - C^T C, & t < 0, \\
K(0) &= 0;
\end{align*} \]
defined on $]-\infty, 0]$. Let $x_0 \in \mathbb{R}^n$; show that $\forall t_1 \leq t_2 \leq 0$: $x_0^T \Pi(t_1, 0, 0)x_0 \geq x_0^T \Pi(t_2, 0, 0)x_0$. 