1. (15 pts) Let \((A, B, C)\) be a minimal LTI realization. Find \(u \in C^0_p([0, \infty[; \mathbb{R}^m)\) such that for
\[
\dot{x}(t) = Ax(t) + Bu(t),
\]
\[
y(t) = Cx(t),
\]
(where \(t \geq 0\)), the cost function
\[
\eta = 2 \int_0^{\infty} u^T(t)u(t)dt + \int_2^{\infty} y^T(t)y(t)dt
\]
is minimized.

2. (15 pts) Let \(x_0 = (x_0, y_0) \in \mathbb{R}^2\), with \(x_0 \neq 0\). Let \(t_0 \in \mathbb{R}\). We want to find a (or the) piecewise \(C^1\) function \(x : [t_0, t_1] \rightarrow \mathbb{R}^2\) (with \(t_1\) undetermined a priori) such that:

- \(x(t_1) \in \{0\} \times \mathbb{R}\) (i.e. the \(x\)-component of \(x\) is zero),
- \(x(t) \notin \{0\} \times \mathbb{R}, \forall t < t_1\) (i.e. \(x\) does not hit the \(y\)-axis of \(\mathbb{R}^2\) before time \(t_1\)),

which minimizes \(\int_{t_0}^{t_1} ||\dot{x}(t)||dt\). In other words, we wish to find a (or the) piecewise \(C^1\) plane curve of minimum Euclidean length which joins \(x_0\) to the \(y\)-axis of \(\mathbb{R}^2\).

(a) Formulate this problem as an optimal control problem.

(b) Compute the value function for this optimal control problem by studying the geometry of the problem, and compute the optimal control and trajectory by using the Dynamic Programming partial differential equation.