1. We have (using the concept of integration):

\[ x(1) = \int_0^{1/3} e^{A(1-\sigma)}b(\sigma)d\sigma = \int_0^{1/3} e^{A(1-\sigma)}b(\sigma)d\sigma + \int_0^{2/3} e^{A(1-\sigma)}b(\sigma)d\sigma + \int_0^{1} e^{A(1-\sigma)}b(\sigma)d\sigma = \int_0^{1/3} e^{A(1-\sigma)}b(\sigma)d\sigma + \int_0^{1/3} e^{A(1-\sigma)-1/3}b(\sigma)d\sigma + \int_0^{1/3} e^{A(1-\sigma)-2/3}b(\sigma)d\sigma \]

\[ = \int_0^{1/3} e^{A(1-\sigma)}b(\sigma)d\sigma + \int_0^{1/3} e^{A(2/3-\sigma)}b(\sigma)d\sigma + \int_0^{1/3} e^{A(1/3-\sigma)}b(\sigma)d\sigma \]

\[ = \int_0^{1/3} (e^{A(1-\sigma)} + e^{A(2/3-\sigma)} + e^{A(1/3-\sigma)})b(\sigma)d\sigma = \int_0^{1/3} (e^{A(2/3)} + e^{A(1/3)})b(\sigma)d\sigma \]

\[ = \int_0^{1/3} e^{A(1/3-\sigma)}(e^{2A/3} + e^{A/3} + I)b(\sigma)d\sigma \]

and this is nothing other than \( z(1/3) \) for the linear time-invariant system defined by \( \dot{z}(t) = Az(t) + (e^{2A/3} + e^{A/3} + I)b(t) \). Hence, we can steer \( x \) from 0 at \( t = 0 \) to \( x_1 \) at \( t = 1 \) if and only if we can steer \( z \) from 0 at \( t = 0 \) to \( z_1 \) at \( t = 1/3 \).

2. (a) Since the family \( \{e^{At_k}\}_{k=0}^{n-1} \) is a linearly independent family of \( M_n(\mathbb{R}) \) by assumption, and since each element of that family is in the linear span of the family \( \{A^k\}_{k=0}^{n-1} \), we deduce that the linear span of the family \( \{e^{At_k}\}_{k=0}^{n-1} \) is at least \( n \)-dimensional. Since this family has \( n \) elements, this linear span is exactly \( n \)-dimensional and the elements in the family are linearly independent. The matrix \( \Gamma \) with \((i,j)\) entry \( \alpha_i(t_j) \) is the expression in the basis \( \{A^k\}_{k=0}^{n-1} \) of the endomorphism (of the linear span of \( \{e^{At_k}\}_{k=0}^{n-1} \)) which maps \( A^k \) into \( e^{At_k} \) for \( k = 0, \ldots, n-1 \). Since this endomorphism is an isomorphism, \( \Gamma \) is non-singular and therefore, \( \det(\Gamma) \neq 0 \).

(b) Consider the linear operator \( L : \mathbb{R}^n \rightarrow \mathbb{R}^n \) defined by \( L : x \mapsto (Ce^{At_0}x, Ce^{At_1}x, \ldots, Ce^{At_{n-1}}x) \). Then \( L(x(0)) \) is the vector of outputs \( y \) observed at times \( t_0, t_1, \ldots, t_{n-1} \). It is possible to recover \( x(0) \) uniquely from \( L(x(0)) \) iff \( L \) is injective, that is, if \( \ker(L) = \{0\} \). Let now \( x \in \ker(L) \). We shall show that \( x = 0 \). This will show that the initial condition \( x(0) \) can be uniquely recovered from the observations \( (Ce^{At_0}x(0), Ce^{At_1}x(0), \ldots, Ce^{At_{n-1}}x(0)) \).

Define \( z_j = CA^jx \) for \( j = 0, \ldots, n-1 \). We have:

\[ x \in \ker(L) \iff Ce^{At_k}x = 0, \quad \forall k = 0, \ldots, n-1, \]

\[ \iff \sum_{i=0}^{n-1} \alpha_i(t_k)CA^i x = 0, \quad \forall k = 0, \ldots, n-1, \]

\[ \iff \sum_{i=0}^{n-1} \alpha_i(t_k)z_i = 0, \quad \forall k = 0, \ldots, n-1, \]

and since the matrix \( \Gamma \) is non-singular, this last line implies that \( z_k = 0 \) for all \( k = 0, \ldots, n-1 \), which in turn implies that \( x \) is in the kernel of the observability matrix; but since \( (C, A) \) is assumed to be an observable pair, the kernel is trivial, and therefore \( x = 0 \).

3. (a) This is a straightforward calculation.
(b) Let $C_{A,B}$ be the controllability matrix for system I, and $C_{TAT^{-1},TB}$ the controllability matrix for system II. We have:

$$\begin{align*}
C_{TAT^{-1},TB} &= \begin{bmatrix} TB & TAT^{-1}TB & (TAT^{-1})^2TB & \cdots & (TAT^{-1})^{n-1}TB \end{bmatrix} \\
&= T \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \\
&= TC_{A,B}
\end{align*}$$

and since $T$ is non-singular, $C_{A,B}$ and $C_{TAT^{-1},TB}$ have equal rank. Hence, system I is controllable iff system II is.

(c) Similar calculation with the observability matrices instead.

(d) Let us show that $\text{Ker}(O_{(C,A)})$ is an $A$–invariant subspace of $\mathbb{R}^n$. Now $z \in \text{Ker}(O_{(C,A)})$ if and only if $O_{(C,A)}z = 0$, and since $O_{(C,A)} = \begin{pmatrix} C & CA & \cdots & CA^{n-1} \end{pmatrix}$, it follows that $z \in \text{Ker}(O_{(C,A)})$ if and only if $CA^kz = 0 \ \forall k \in \{0,1,\cdots,n-1\}$.

Let now $z \in \text{Ker}(O_{(C,A)})$; we wish to show that $Az \in \text{Ker}(O_{(C,A)})$, i.e. that $CA^k(Az) = 0 \ \forall k \in \{0,1,\cdots,n-1\}$, i.e. that $CA^kz = 0 \ \forall k \in \{1,\cdots,n\}$. Since we have assumed that $z \in \text{Ker}(O_{(C,A)})$, we already have that $CA^kz = 0 \ \forall k \in \{0,1,\cdots,n-1\}$. We need only show therefore that $CA^nz = 0$. Since $A \in M_n(\mathbb{R})$, it follows from the Cayley-Hamilton theorem that there exist $\alpha_0, \cdots, \alpha_{n-1} \in \mathbb{R}$ such that $A^n = \alpha_0I + \alpha_1A + \alpha_2A^2 + \cdots + \alpha_{n-1}A^{n-1}$.

Hence,

$$\begin{align*}
CA^nz &= C(\alpha_0I + \alpha_1A + \alpha_2A^2 + \cdots + \alpha_{n-1}A^{n-1})z \\
&= \alpha_0Cz + \alpha_1CAz + \alpha_2CA^2z + \cdots + \alpha_{n-1}CA^{n-1}z \\
&= 0,
\end{align*}$$

and this proves the desired result.

4. The controllability matrix for $\dot{x}(t) = Ax(t) + Bu(t)$ is given by $C_{(A,B)} = \begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix}$, and its transpose is given by $C^T_{(A,B)} = \begin{pmatrix} B^T & B^TA^T & \cdots & B^T(A^T)^{n-1} \end{pmatrix}$, which is nothing other than the observability matrix for the system

$$\begin{align*}
\dot{x}(t) &= A^Tx(t), \\
y(t) &= B^Tx(t).
\end{align*}$$

Therefore, the controllability matrix of the first system has the same rank as the observability matrix of the second system. Therefore, the first system is controllable iff the second system is observable.