1. (10 pts) Consider the linear time-varying control system

\[ \begin{align*}
\dot{x}(t) &= Ax(t) + (\sin t)Bu(t) \\
\dot{z}(t) &= Az(t) + (\cos t)Bu(t) \\
y(t) &= (\sin t)Cx(t) + (\cos t)Cz(t)
\end{align*} \]

where \( x(t), z(t) \in \mathbb{R}^n \) for all \( t \in \mathbb{R} \), and \( A, B, C \) are given matrices. Show that the weighting pattern for this system is of the form \( T : (t, \sigma) \mapsto T(t, \sigma) = T(t - \sigma) \) for some function \( T \). (Note that the state variables are \( x \) and \( z \); as always, the input is \( u \) and the output, \( y \).)

2. (20 pts) Consider the linear time-invariant system

\[ \begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t),
\end{align*} \]

where \( A \in M_n(\mathbb{R}), B \) is an \( n \times m \) real matrix, and \( C \) is a \( p \times n \) real matrix. Denote by \( C_{A,B} \) the controllability matrix of the pair \( (A, B) \), and by \( O_{C,A} \) the observability matrix of the pair \( (C, A) \). Recall that \( \text{Im}(C_{A,B}) \) and \( \text{Ker}(O_{C,A}) \) are \( A \)-invariant subspaces of \( \mathbb{R}^n \). This implies in particular that \( \text{Ker}(O_{C,A}) \cap \text{Im}(C_{A,B}) \) is an \( A \)-invariant subspace of \( \mathbb{R}^n \). Show that there exists a non-singular transformation \( T : \mathbb{R}^n \to \mathbb{R}^n \) such that

\[ T^{-1}AT = \begin{pmatrix} A_1 & 0 & A_6 & 0 \\
A_2 & A_3 & A_4 & A_5 \\
0 & 0 & A_7 & 0 \\
0 & 0 & A_8 & A_9 \end{pmatrix}, \quad T^{-1}B = \begin{pmatrix} B_1 \\
B_2 \\
0 \\
0 \end{pmatrix}, \quad CT = \begin{pmatrix} C_1 & 0 & C_2 & 0 \end{pmatrix} \]

and such that

- the pair \( (C_1, A_1) \) is observable,
- the pair \( (A_1, B_1) \) is controllable,
- the pair \( \begin{pmatrix} A_1 & 0 \\
A_2 & A_3 \end{pmatrix}, \begin{pmatrix} B_1 \\
B_2 \end{pmatrix} \) is controllable,
- the pair \( \begin{pmatrix} C_1 & C_2 \end{pmatrix}, \begin{pmatrix} A_1 & A_6 \\
0 & A_7 \end{pmatrix} \) is observable.

(Hint: \( \mathbb{R}^n \) can be written as the direct sum of \( \text{Im}(C_{A,B}) \) and a complementary subspace; it can also be written as a direct sum of \( \text{Ker}(O_{C,A}) \) and a complementary subspace. Consider the pairwise intersections of these subspaces.)

3. (20 pts) Let \( Q : t \mapsto Q(t) \) be a mapping from \( \mathbb{R}^+ \) to the vector space \( M_{p,m}(\mathbb{R}) \) of real \( p \times m \) matrices. Assume \( Q \) is continuous (equivalently, each of the entries of \( Q \) is a continuous real-valued function). Show that if for all continuous functions \( u : \mathbb{R}^+ \to \mathbb{R}^m \) and for all \( t \geq 0 \) we have \( \int_0^t Q(\tau)u(\tau)d\tau = 0 \), then necessarily, \( Q(t) = 0 \) \( \forall t \geq 0 \). (Note: We will use this result in our characterization of equivalent LTI realizations.)

4. (20 pts) Let \( A \in M_n(\mathbb{R}) \) and \( B \in M_{n,m}(\mathbb{R}) \) (i.e. \( B \) is a real \( n \times m \) matrix). Show that if for all \( \lambda \in \mathbb{C} \) we have \( \text{rank}((A - \lambda I) \ B) = n \), then the pair \( (A, B) \) is controllable (here \( I \) denotes the \( n \times n \) identity matrix). (Hint: Show, using the controllability normal form, that if \( (A, B) \) is not controllable, then \( \exists \lambda \in \mathbb{C} \) such that \( [(A - \lambda I) \ B] \) has rank \( < n \).)

5. (10 pts) Let \( A \) be a non-singular \( n \times n \) real matrix with distinct eigenvalues. Show that there exists a matrix \( C \), possibly complex, such that \( A = e^C \).

6. (10 pts) Let \( A \) be a real \( m \times n \) matrix. The rank \( \text{rank}(A) \) of \( A \) is defined as \( \text{dim}((\text{Im}(A)) \). Prove that \( \text{rank}(A) = \text{rank}(A^T) \).