Solutions to Problem Set #6 Monday, 7/11/2016

Queen’s University - Math 430/830 Fall 2016

1. (Note: identity matrices will all be denoted by \( \mathbf{I} \) irrespective of their dimension).

(a) Let \( \mathbf{G} = \mathbf{G}_1 + 2\mathbf{G}_2 \). Then, \( \forall s \) in an appropriate open domain of \( \mathbb{C} \), we have:

\[
\mathbf{G}(s) = \{ \mathbf{C}_1(s\mathbf{I} - \mathbf{A}_1)^{-1}\mathbf{B}_1 + \mathbf{D}_1 \} + 2 \{ \mathbf{C}_2(s\mathbf{I} - \mathbf{A}_2)^{-1}\mathbf{B}_2 + \mathbf{D}_2 \}
\]

\[
= ( \mathbf{C}_1 \ 2\mathbf{C}_2 ) \left( \begin{array}{c}
(s\mathbf{I} - \mathbf{A}_1)^{-1} \\
(s\mathbf{I} - \mathbf{A}_2)^{-1}
\end{array} \right) \left( \begin{array}{c}
\mathbf{B}_1 \\
\mathbf{B}_2
\end{array} \right) + (\mathbf{D}_1 + 2\mathbf{D}_2)
\]

and therefore

\[
\left[ \begin{array}{c}
\mathbf{A}_1 \\
\mathbf{B}_1
\end{array} \right], \left[ \begin{array}{c}
\mathbf{A}_2 \\
\mathbf{B}_2
\end{array} \right], \left[ \begin{array}{c}
\mathbf{C}_1 \\
2\mathbf{C}_2
\end{array} \right], \mathbf{D}_1 + 2\mathbf{D}_2
\]

is a realization of \( \mathbf{G} \).

(b) Let now \( \mathbf{G} = \mathbf{G}_1\mathbf{G}_2 \). Then \( \forall s \) in an appropriate open domain of \( \mathbb{C} \), we have:

\[
\mathbf{G}(s) = \mathbf{C}_1(s\mathbf{I} - \mathbf{A}_1)^{-1}\mathbf{B}_1\mathbf{C}_2(s\mathbf{I} - \mathbf{A}_2)^{-1}\mathbf{B}_2 + \mathbf{C}_1(s\mathbf{I} - \mathbf{A}_1)^{-1}\mathbf{B}_1\mathbf{D}_2 + \mathbf{D}_1\mathbf{C}_2(s\mathbf{I} - \mathbf{A}_2)^{-1}\mathbf{B}_2 + \mathbf{D}_1\mathbf{D}_2
\]

The difficulty here comes from the first term, which is a product of matrices involving both \((s\mathbf{I} - \mathbf{A}_1)^{-1}\) and \((s\mathbf{I} - \mathbf{A}_2)^{-1}\). Could we get this kind of term by choosing \( \mathbf{A} \) of the form \( \left( \begin{array}{c}
\mathbf{A}_1 \\
\mathbf{A}_2
\end{array} \right) \)? Well, in that case, we will have:

\[
(s\mathbf{I} - \mathbf{A})^{-1} = (s\mathbf{I} - \left( \begin{array}{c}
\mathbf{A}_1 \\
\mathbf{A}_2
\end{array} \right))^{-1} = \left( \begin{array}{c}
(s\mathbf{I} - \mathbf{A}_1)^{-1} \\
(s\mathbf{I} - \mathbf{A}_2)^{-1}
\end{array} \right)^{-1} = \left( \begin{array}{c}
(s\mathbf{I} - \mathbf{A}_1)^{-1} \\
(s\mathbf{I} - \mathbf{A}_2)^{-1}
\end{array} \right)
\]

Multiplying this matrix to the left by \( \mathbf{C} \) and to the right by \( \mathbf{B} \) will generate only linear combinations of \((s\mathbf{I} - \mathbf{A}_1)^{-1}\) and \((s\mathbf{I} - \mathbf{A}_2)^{-1}\), and will not give any term containing the product of these. Let us now choose a slightly more complicated \( \mathbf{A} \); more precisely, let us take \( \mathbf{A} \) to be

\[
\mathbf{A} = \left( \begin{array}{c}
\mathbf{A}_1 \\
\mathbf{0}
\end{array} \right).
\]

Then, we have:

\[
(s\mathbf{I} - \mathbf{A}) = \left( \begin{array}{c}
(s\mathbf{I} - \mathbf{A}_1) \\
0
\end{array} \right).
\]

Since this matrix is block-upper-triangular, its inverse can be written as

\[
(s\mathbf{I} - \mathbf{A})^{-1} = \left( \begin{array}{c}
(s\mathbf{I} - \mathbf{A}_1)^{-1} \\
0
\end{array} \right),
\]

for some yet to be determined \( \mathbf{Q} \). How do we determine \( \mathbf{Q} \)? By using the fact that the last two matrices are inverses of one another, i.e. their product is the identity matrix. That is,

\[
\left( \begin{array}{c}
(s\mathbf{I} - \mathbf{A}_1) \\
0
\end{array} \right) \left( \begin{array}{c}
(s\mathbf{I} - \mathbf{A}_1)^{-1} \\
0
\end{array} \right) = \left( \begin{array}{c}
\mathbf{I} \\
\mathbf{0}
\end{array} \right),
\]

\[
\left( \begin{array}{c}
\mathbf{A}_1 \\
\mathbf{A}_2
\end{array} \right) \left( \begin{array}{c}
\mathbf{A}_1 \\
\mathbf{A}_2
\end{array} \right) = \left( \begin{array}{c}
\mathbf{I} \\
\mathbf{0}
\end{array} \right).
\]

which yields
\[
\begin{pmatrix}
I & (sI - A_1)Q - P(sI - A_2)^{-1} \\
0 & I
\end{pmatrix} = \begin{pmatrix}
I & 0 \\
0 & I
\end{pmatrix},
\]
which implies
\[
(sI - A_1)Q - P(sI - A_2)^{-1} = 0,
\]
from which we obtain:
\[
Q = (sI - A_1)^{-1}P(sI - A_2)^{-1}.
\]
Now this is exactly the kind of term we were after. Choosing \(P = B_1C_2\) then gives:
\[
(sI - \begin{pmatrix} A_1 & B_1C_2 \\ 0 & A_2 \end{pmatrix})^{-1} = \begin{pmatrix} (sI - A_1)^{-1} & (sI - A_1)^{-1}B_1C_2(sI - A_2)^{-1} \\ 0 & (sI - A_2)^{-1} \end{pmatrix}
\]
Choosing the appropriate \(B, C, D\) matrices is now immediate, and we can verify easily that
\[
\begin{pmatrix} C_1 & D_1C_2 \\ 0 & B_2 \end{pmatrix} \begin{pmatrix} (sI - A_1)^{-1} & (sI - A_1)^{-1}B_1C_2(sI - A_2)^{-1} \\ 0 & (sI - A_2)^{-1} \end{pmatrix} \begin{pmatrix} B_1D_2 \\ B_2 \end{pmatrix} + D_1D_2 = G(s).
\]
Hence,
\[
\begin{pmatrix} A_1 & B_1C_2 \\ 0 & A_2 \end{pmatrix}, \begin{pmatrix} B_1D_2 \\ B_2 \end{pmatrix}, \begin{pmatrix} C_1 & D_1C_2 \\ D_1D_2 \end{pmatrix}
\]
is a realization of \(G\).

(a) Consider the transfer function \(s \mapsto G_1(s) = \begin{pmatrix} \frac{2}{s} & 0 \\ \frac{1}{s^2} & \frac{1}{s} \end{pmatrix}\). We have to find matrices \(A, B, C, D\) such that \(C(sI - A)^{-1}B + D = G_1(s)\). Note first that \(D = 0\) since each entry is a strictly proper rational function of \(s\) (i.e. for each entry, the degree of the numerator is strictly less than the degree of the denominator). Note also that since each of the denominators in each entry must be a factor in the characteristic polynomial of \(A\), the characteristic polynomial must have a factor equal to \(s^2\). This implies, that at the very least, \(A\) must be a \(2 \times 2\) matrix with 0 being an eigenvalue of \(A\) with multiplicity at least 2. Let us begin with the smallest possible dimension, namely 2; up to a similarity transformation, there are not too many choices. If \(A\) is diagonal with 0 on the diagonal, then \(A\) is the 0 matrix. In this case, \((sI - A)^{-1}\) will have no \(\frac{1}{s^2}\) term, as is easy to verify. The next possibility is to have \(A\) of the form \(A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\) (the 1 in the upper right corner can be replaced with any non-zero number; we take 1 for simplicity). With that choice of \(A\), we have \((sI - A)^{-1} = \begin{pmatrix} \frac{1}{s} & \frac{1}{s} \\ 0 & \frac{1}{s} \end{pmatrix}\). Choosing \(B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\) and \(C = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}\) (and \(D = 0\) as indicated above) yields the desired realization. Note that \(B\) has rank 2. Hence the \(2 \times 4\) controllability matrix \([B \ AB]\) will also have rank 2, that is, the realization \([A, B, C, D]\) is controllable. Similarly, \(C\) has rank 2. Hence the \(4 \times 2\) observability matrix \(\begin{pmatrix} C \\ CA \end{pmatrix}\) of this realization has rank 2, that is, the realization \([A, B, C, D]\) is observable. Hence, this realization is minimal. Note that we could also have deduced minimality in this case by observing that any realization of \(G_1\) has to have dimensionality at least 2 (since there is no way we could have a \(\frac{1}{s^2}\) term otherwise), and ours has exactly that dimensionality, hence must be minimal.
Consider now the transfer function $s \mapsto G_2(s) = \begin{pmatrix} 0 & \frac{3}{s} \\ \frac{1}{s} & 1 \end{pmatrix}$. Here again, the same observations as above do hold. However, experimenting with $A$ the $0$ matrix and $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ quickly convinces us that there may be no $2 \times 2$ realization for $s \mapsto G_2$. We then try to find a realization of degree 3. Modulo similarity transformations, there is a larger choice for $A$. Here again, $A = 0$ does not work since $(sI - A)^{-1}$ will have entries which are either 0 or $\frac{1}{s}$. Some trial and error with various simple choices of $A$ strictly upper triangular (implying eigenvalues zero; note also what matters here is the conjugacy class of $A$) leads to $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. This yields $(sI - A)^{-1} = \begin{pmatrix} \frac{1}{s} & 0 & \frac{1}{s} \\ 0 & \frac{1}{s} & 0 \\ 0 & 0 & \frac{1}{s} \end{pmatrix}$. Choosing now $B = \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ yields $C(sI - A)^{-1}B = G_1(s)$. Hence, with $D = 0$, $[A, B, C, D]$ is a realization of the transfer function $G_1$. Since $A \in M_3(\mathbb{R})$, this is a realization of degree 3. To verify that we have a minimal realization, we have to verify that this realization is both controllable and observable. The controllability matrix $[B \ AB \ A^2B]$ is given by $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, which is clearly of rank 3; hence the realization is controllable. Similarly, the observability matrix $\begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix}$ is given by $\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$, which is clearly of rank 3; hence the realization is observable. Being both controllable and observable, this realization of $G_1$ is minimal.

2. Note first that if $[A^T, C^T, B^T]$ is indeed a realization of $\hat{G}$, it is necessarily minimal, since the realization $[A, B, C]$ of $\hat{G}$ is assumed minimal and the matrices $A$ and $A^T$ have the same dimension. So we need only show that $[A^T, C^T, B^T]$ is a realization of $\hat{G}$. We have:

$$\hat{G}(s) = \hat{G}^T(s) = (C(sI - A)^{-1}B)^T = B^T ((sI - A)^{-1})^T C^T = B^T(sI - A^T)^{-1}C^T,$$

which shows that $[A^T, C^T, B^T]$ is a realization of $\hat{G}$.

3. Let $A \in M_n(\mathbb{R})$.

(a) By Jordan’s Normal Form Theorem, there exists $P \in M_n(\mathbb{C})$ with $P$ non-singular, such that

$$PAP^{-1} = \begin{pmatrix} J_1 \\ & J_2 \\ & & \ddots \\ & & & J_r \end{pmatrix},$$

where each Jordan block $J_i$ is an $n_i \times n_i$ matrix of the form $J_i = (\lambda_i)$ for $n_i = 1$ and, for $n_i \geq 2$, of the form...
$J_i = \lambda_i I_i + N_i$, with $\lambda_i \in \mathbb{C}$, and where $I_i$ denotes the $n_i \times n_i$ identity matrix and $N_i$ the nilpotent $n_i \times n_i$ matrix

$$
N_i = \begin{pmatrix}
0 & 1 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & \cdots \\
\vdots & \vdots & \ddots & \vdots & \ddots \\
0 & \cdots & 0 & 1 & \cdots \\
0 & \cdots & \cdots & 0 & 0
\end{pmatrix}
$$

Note that for $n_i \geq 2$, $N_i^{n_i} = 0$.

Let $\Lambda = P \Lambda P^{-1}$. Then, $\forall t \in \mathbb{R}$:

$$
e^{A_t} = e^{P^{-1} \Lambda t} P = P^{-1} e^{\Lambda t} P = P^{-1} \begin{pmatrix}
e^{J_1 t} & e^{J_2 t} & \cdots & e^{J_r t} \end{pmatrix} P$$

For each $i \in 1, \cdots, r$, we have:

$$e^{J_i t} = e^{(\lambda_i I_i + N_i) t} = e^{(\lambda_i I_i + N_i t)} = e^{\lambda_i I_i} e^{N_i t},$$

since the matrices $I_i$ and $N_i$ commute, and hence,

$$e^{J_i t} = e^{\lambda_i t} e^{N_i t}.$$ 

It follows from this that if $\text{Re}(\lambda_i) < 0$ then $\lim_{t \to \infty} e^{J_i t} = 0$. Hence, if all the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_r$ have real part < 0, then $\lim_{t \to \infty} e^{A t} = 0$, which implies

$$\lim_{t \to \infty} e^{A t} = \lim_{t \to \infty} (P^{-1} e^{\Lambda t} P) = P^{-1} \left( \lim_{t \to \infty} e^{A t} \right) P = 0.$$

As a result, if all the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_r$ have real part < 0, then, $\forall x_0 \in \mathbb{R}^n$, we have:

$$\lim_{t \to \infty} (e^{A t} x_0) = \left( \lim_{t \to \infty} e^{A t} \right) x_0 = 0.$$

(b) Assume now that one of the eigenvalues of $A$, say $\lambda$, has real part $\geq 0$; we shall show that this implies the existence of some $x_0 \in \mathbb{R}^n$ such that $e^{A t} x_0$ does not go to 0 as $t \to \infty$. Let $v \in \mathbb{C}^n$, $v \neq 0$, be an eigenvector of $A$ corresponding to eigenvalue $\lambda$. We then have $A v = \lambda v$. Since $A$ is a real matrix, we obtain upon conjugation:

$$\overline{A v} = \overline{\lambda v} = \lambda v;$$

(Note: By $\overline{v}$ we mean the vector obtained by conjugating each entry of $v$; same for $A$ and $\overline{A}$). On the other hand, we have:

$$\overline{A v} = \overline{\lambda v} = \overline{\lambda v},$$

and therefore,

$$A \overline{v} = \overline{\lambda v},$$

which shows that $\overline{v}$ is an eigenvector of $A$ corresponding to eigenvalue $\overline{\lambda}$. Let now $x_0 = \frac{v + \overline{v}}{2}$. Clearly, all entries of $x_0$ are real, i.e. $x_0 \in \mathbb{R}^n$. Since $A v = \lambda v$, we obtain, $\forall t \in \mathbb{R}$:

$$e^{A t} v = e^{\lambda t} v.$$
Similarly, from $A\bar{v} = \lambda \bar{v}$, we obtain, $\forall t \in \mathbb{R}$:

$$e^{A t} \bar{v} = e^{\lambda t} \bar{v}.$$  

Hence, $\forall t \in \mathbb{R}$:

$$e^{A t} x_0 = \frac{1}{2}(e^{\lambda t} v + e^{\bar{\lambda} t} \bar{v}).$$

Let now $\lambda = \sigma + i\omega$, with $\sigma \geq 0$. Since $v \neq 0$, it must have at least one non-zero entry. Assume the $k$th entry $v_k$ of $v$ is non-zero, and let that entry be $v_k = re^{i\theta}$, for some $r > 0$ and $\theta \in \mathbb{R}$. The $k$th entry of $e^{A t} x_0$ is then given by

$$\frac{1}{2}(e^{\lambda t} v_k + e^{\bar{\lambda} t} \bar{v}_k) = \Re(e^{\lambda t} v_k) = re^{\sigma t} \cos(\omega t + \theta),$$

and since $\sigma \geq 0$, we have $\lim_{t \to \infty} re^{\sigma t} \cos(\omega t + \theta) \neq 0$, which implies $\lim_{t \to \infty} e^{A t} x_0 \neq 0$.

4. (a) We have:

$$\text{Im}(V_\lambda) = \{V_\lambda x \mid x \in \mathbb{R}^{n+m}\} = \{(A - \lambda I)y + Bz \mid y \in \mathbb{R}^n, z \in \mathbb{R}^m\}.$$  

$\forall v \in V_\lambda, \exists y \in \mathbb{R}^n, \exists z \in \mathbb{R}^m$ such that $v = (A - \lambda I)y + Bz$; hence,

$$Av = A(A - \lambda I)y + ABz = (A - \lambda I)(Ay + Bz) + B(\lambda z),$$

which shows that $Av \in \text{Im}(V_\lambda)$. Hence, $V_\lambda$ is an $A$–invariant subspace of $\mathbb{R}^n$.

(b) Clearly, $\forall \lambda \in \mathbb{R}$, we have $\text{Im}(B) \subset \text{Im}(V_\lambda)$. Hence, $\text{Im}(AB) = A(\text{Im}(B)) \subset A(\text{Im}(V_\lambda)) = \text{Im}(V_\lambda)$ by $A$–invariance of $V_\lambda$, i.e. $\text{Im}(AB) \subset \text{Im}(V_\lambda)$ (with – some abuse of notation – the symbol “$\subset$” denoting here a subspace relation and not a mere inclusion). Continuing in this way, we obtain $\text{Im}(A^kB) \subset \text{Im}(V_\lambda), \forall k \in \mathbb{N}$, and, as a result, we have $\text{Im}(C_{A,B}) \subset \text{Im}(V_\lambda), \forall \lambda \in \mathbb{R}$.

(c) Note that (b) implies that $\forall \lambda \in \mathbb{R}$, $\text{rank}(C_{A,B}) \leq \text{rank}(V_\lambda)$. Suppose now that for some $\lambda \in \mathbb{R}$ we have $\text{rank}(V_\lambda) < n$; it then follows that $\text{rank}(C_{A,B}) < n$, which then implies that the pair $(A, B)$ is not controllable.