Queen’s University - Math 430/830  
Problem Set #9  
Fall 2016  

1. (20 pts) Let \( T > 0 \), and consider the problem of minimizing \( \eta = \int_0^T (\dot{x}^2(t) - x^2(t)) \, dt \) over all \( C^1 \) functions \( x : [0, T] \to \mathbb{R} \) with \( x(0) = 1 \).

   (a) Formulate this problem as an optimal control problem.

   (b) Show that the associated Riccati equation has a \( C^1 \) solution on \( [0, T] \) only for \( 0 < T < \frac{\pi}{2} \).

   (c) Solve the optimal control problem for \( 0 < T < \frac{\pi}{2} \), i.e. compute the optimal control, the optimal trajectory, and the corresponding value of \( \eta \).

   (d) To understand what happens when \( T = \frac{\pi}{2} \), consider the family \( \{x_\mu\}_{\mu \in \mathbb{R}} \) of \( C^1 \) functions on \( [0, T] \) defined by \( t \mapsto x_\mu(t) = \cos t - \mu \sin t \); note that \( \forall \mu \in \mathbb{R} \) we have \( x_\mu(0) = 1 \), i.e. this family of functions satisfies the desired boundary condition. \( \forall \mu \in \mathbb{R} \), let \( \eta_\mu = \int_0^T (\dot{x}_\mu^2(t) - x_\mu^2(t)) \, dt \). By computing and studying \( \eta_\mu \), explain why the cases \( 0 < T < \frac{\pi}{2} \) and \( T = \frac{\pi}{2} \) are qualitatively different for the optimal control problem.

2. (20 pts) Let \( T > 0 \) and \( \alpha \in \mathbb{R} \), and consider the problem of minimizing \( \eta = \int_0^T (\dot{x}^2(t) + x^2(t)) \, dt + \alpha x^2(T) \) over all \( C^1 \) functions \( x : [0, T] \to \mathbb{R} \) with \( x(0) = 1 \).

   (a) Formulate this problem as an optimal control problem.

   (b) For which values of \( \alpha \) does the associated Riccati equation have a \( C^1 \) solution on \( [0, T] \) ? Solve the Riccati equation for those values of \( \alpha \).

   (c) Solve the optimal control problem for the values of \( \alpha \) found in (b), i.e. compute the optimal control, the optimal trajectory, and the corresponding value of \( \eta \).

3. (10 pts) Consider the linear time-varying system \( \dot{x}(t) = A(t)x(t) + B(t)u(t) \) and suppose we wish to keep the trajectory \( t \mapsto x(t) \) close to the trajectory \( t \mapsto y(t) \) on the interval \( [0, T] \) by computing the control \( u : [0, T] \to \mathbb{R}^m \) which minimizes the cost functional \( \eta \), given by:

\[
\eta(u) = \int_0^T \left( u^T(t)u(t) + [x(t) - y(t)]^T L(t)[x(t) - y(t)] \right) \, dt, \quad \forall u \in C^0_p([0, T]; \mathbb{R}^m)
\]

Assuming that \( y \) satisfies the linear homogeneous equation \( \dot{y}(t) = A(t)y(t) \), show that the control \( u \) which minimizes \( \eta \) is of the form \( u(t) = F(t)x(t) + g(t) \), and compute \( F \) and \( g \).

4. (10 pts) Find the minimum value of \( \int_0^\pi \dot{x}^2(t) \, dt \) on the set of all \( C^1 \) functions \( x : [0, \pi] \to \mathbb{R} \) satisfying \( x(0) = 1 \) and \( x(\pi) = 0 \).