1. The Taylor expansion of $F$ (evaluated at $t = 1$) around $t = 0$ yields:

$$F(1) = F(0) + F'(0) + (\text{remainder})$$

By the chain rule,

$$F'(0) = \frac{d}{dt} t=0(F(t)) = \frac{d}{dt} t=0(f(a + t(b-a))) = Df|_a \cdot (b-a),$$

and therefore, using the fact that $F(1) = f(b)$, we obtain:

$$f(b) = f(a) + Df|_a \cdot (b-a) + (\text{remainder}).$$

2. Part (a) is a simple calculation. For part (b), the linearization around the nominal trajectory is obtained by first putting the system in first-order form. Let then $x_1 = x$, $x_2 = \dot{x} = \dot{x}_1$. We have:

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} x_2(t) \\ -x_1^2(t) - (\sin t)x_2(t) + 1 + u(t) \end{pmatrix} = \begin{pmatrix} f_1(x_1(t), x_2(t), u(t)) \\ f_2(x_1(t), x_2(t), u(t)) \end{pmatrix},$$

and

$$y(t) = x_2^2(t) = g(x_1(t), x_2(t), u(t)).$$

The linearization of this nonlinear system around the given trajectory is defined to be the linear system

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{pmatrix} u(t),$$

and

$$y(t) = \begin{pmatrix} \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \frac{\partial g}{\partial u} u(t),$$

where all matrices are evaluated at the given trajectory of the nonlinear system. Evaluating these matrices at the given trajectory, we obtain the linear time-varying system:

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2\cos t & -\sin t \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t),$$

$$y(t) = \begin{pmatrix} 0 & -2\sin t \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}.$$

3. Let us define $T : \mathbb{R} \to \mathbb{R}$ by $T(x) = 1$ for all $x < 0$, and $T(x) = x + e^{-x}$, $\forall x \geq 0$. Note that $T$ is continuous on $\mathbb{R}$. It is immediately verified that $|T'(x)| = |1 - e^{-x}| < 1$ for all $x \geq 0$, and therefore, by the mean value theorem, $|T(x) - T(y)| < |x - y|$, for all $x, y \geq 0, x \neq y$. Clearly, $T$ being constant on $\mathbb{R}^-$, we have $|T(x) - T(y)| < |x - y|$, for all $x, y \leq 0, x \neq y$. It remains to verify the case where $x < 0$ and $y > 0$: we have then $|T(x) - T(y)| = |T(0) - T(y)| < |y| < |x - y|$. We conclude: For all $x, y \in \mathbb{R}$, with $x \neq y$: $|T(x) - T(y)| < |x - y|$. We now show that $T$ has no fixed point. For $x < 0$, $T(x) = 1$, and hence $T(x) \neq x$. For $x \geq 0$, $e^{-x} > 0$, which implies $T(x) \neq x$. Hence, $\forall x \in \mathbb{R}$: $T(x) \neq x$. 
4. With the choice of control function $u(t) = 0 \forall t \geq 0$, the state function $t \mapsto x(t)$ of the control system must satisfy the ODE $\dot{x} = 4x^2$ with initial condition $x(0) = 1$; a solution to this ODE is given by $z(t) = \frac{1}{1-4t}$, $\forall t \in ]-\infty, 1/4]$, as can be seen by directly integrating this ODE (which is separable). Assume now there existed some other function $w : \mathbb{R} \to \mathbb{R}$, $C^1$ on $\mathbb{R}$ and solution to the same ODE with the same initial condition (i.e. $w(0) = 1$). We wish to prove that this leads to a contradiction. This will show that the given ODE (with the given initial condition) has no solution that is $C^1$ on all of $\mathbb{R}$.

Let then $0 < a < 1$ arbitrary. Both $z$ and $w$ are $C^1$, hence continuous, on $[-a, a]$. Hence there exists $r_a > 0$ large enough such that $\forall t \in [-a, a] : z(t), w(t) \in B(1; r_a)$, where $B(1; r_a)$ is the closed ball in $\mathbb{R}$ of center 1 and radius $r_a$. Let $U$ be the open subset $U = ]-1 - 2r_a, 1 + 2r_a[; clearly $B(1; r_a) \subset U$; furthermore the function $f : x \mapsto f(x) = x^2$ is Lipschitz on $U$ with constant $2(1 + 2r_a)$. Applying the uniqueness part of Picard’s theorem, we obtain that $z$ and $w$ must coincide on the interval $[-a, a]$; since this holds for any $a \in [0, 1/4[$, we have that $z$ and $w$ must coincide on the interval $]-1/4, 1/4[$. In particular, since $z(t) \to \infty$ as $t \to 1/4$ with $t < 1/4$, we must have also $w(t) \to \infty$ as $t \to 1/4$ with $t < 1/4$. Hence $w$ is not continuous at $1/4$. This contradicts our assumption that $w$ is $C^1$ on $\mathbb{R}$.

5. Part (a) of the problem is a simple calculation. For part (b), note that with $u(t) = 0 \forall t \geq 0$, the state function $t \mapsto x(t)$ of the control system satisfies an ODE of the form $\dot{x}(t) = f(x(t))$, where $f$ is given by $f(x) = 4x^{3/4}$ for all $x \in \mathbb{R}$. Note that

$$\frac{f(x) - f(0)}{x - 0} = 4x^{-1/4} \to \infty \text{ as } x \to 0.$$ 

This shows that the restriction of $f$ to any neighborhood of 0 is non Lipschitz. Picard’s theorem can therefore not be applied to assert existence and unicity of a local solution with initial value $x = 0$. Here we have existence of a solution, but not uniqueness.