1. Let \( t_0, t_1 \in \mathbb{R} \) with \( t_0 < t_1 \). Recall that \( \mathcal{F}([t_0,t_1];\mathbb{R}^m) \) (equipped with the usual operations of additions of functions and multiplication by a scalar) denotes the \( \mathbb{R} \)–vector space of \( \mathbb{R}^m \)–valued functions on \([t_0,t_1] \), and that \( \mathcal{C}^0_0([t_0,t_1];\mathbb{R}^m) \) denotes the set of \( \mathbb{R}^m \)–valued piecewise-continuous functions on \([t_0,t_1] \) (as defined in class).

(i) Show that \( \mathcal{C}^0_0([t_0,t_1];\mathbb{R}^m) \) is a vector subspace of the \( \mathbb{R} \)–vector space \( \mathcal{F}([t_0,t_1];\mathbb{R}^m) \) (and hence is itself a \( \mathbb{R} \)–vector space, with the vector space operations inherited from \( \mathcal{F}([t_0,t_1];\mathbb{R}^m) \)).

(ii) Show that the \( \mathbb{R} \)–vector space \( \mathcal{C}^0_0([t_0,t_1];\mathbb{R}^m) \) is infinite-dimensional.

2. In this problem, we will prove a linear algebra result that we used in our study of controllability for linear time-varying systems.

(a) Let \( V \) be a real vector space with inner product \( \langle \cdot, \cdot \rangle \). Let \( L : V \to V \) be a (continuous) linear mapping, and assume it is symmetric, i.e. \( \forall u,v \in V : \langle Lu,v \rangle = \langle u,Lv \rangle \). Show that we then have \( \text{Im}(L) \perp \text{Ker}(L) \), where \( \perp \) denotes orthogonality with respect to the inner product \( \langle \cdot, \cdot \rangle \).

(b) Continuing (a), show that if \( V \) is additionally finite-dimensional, then \( V = \text{Im}(L) \oplus \text{Ker}(L) \).

(c) Construct a real \( 2 \times 2 \) matrix \( M \) and a vector \( x_1 \in \mathbb{R}^2 \) such that \( x_1 \notin \text{Im}(M) \) and such that \( \forall x_2 \in \text{Ker}(L) \) we have \( x_2^T x_1 = 0 \). (Note that \( M \) must necessarily be non-symmetric. This shows that in our study of controllability, symmetry of the controllability gramian \( W(t_0,t_1) \) played an essential role.)

3. Consider the Linear Time Varying system given by

\[
\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad t \in J,
\]

where \( J \subset \mathbb{R} \) is some interval of \( \mathbb{R} \), and \( A, B \) are continuous on \( J \). Recall the controllability Gramian \( W \) of this system, defined for the pair \((t_0,t_1)\) (where \( t_0, t_1 \in J, \ t_1 > t_0 \)) by:

\[
W(t_0,t_1) = \int_{t_0}^{t_1} \Phi(t_0,\tau)B(\tau)B^T(\tau)\Phi^T(t_0,\tau)d\tau.
\]

Using the algebraic matrix equation satisfied by the controllability Gramian, show that if for some \( t_0, t_1 \in J \) with \( t_0 < t_1 \) and for each \( x_0, x_1 \in \mathbb{R}^n \), there exists a continuous control on \([t_0,t_1]\) that steers the system from \( x_0 \) at \( t_0 \) to \( x_1 \) at \( t_1 \), then for each \( t_2 \in J \) with \( t_2 > t_1 \) and for each \( x_0, x_2 \in \mathbb{R}^n \), there exists a continuous control on \([t_0,t_2]\) that steers the system from \( x_0 \) at \( t_0 \) to \( x_2 \) at \( t_2 \).
4. Consider the linear time-invariant control system given by \( \dot{x}(t) = g(t) (Ax(t) + Bu(t)), \) \( t \in \mathbb{R}, \) where \( x(t) \in \mathbb{R}^n \) and \( u(t) \in \mathbb{R}^m \) for all \( t \in \mathbb{R}. \) \( A \) is an \( n \times n \) real matrix, and \( B \) an \( n \times m \) real matrix. Assume also that \( g \) is continuous on \( \mathbb{R} \) and that there exist \( \alpha, \beta > 0 \) such that \( \forall t \in \mathbb{R}, 0 < \alpha \leq g(t) \leq \beta < \infty. \) Show that if \( \text{rank} [B AB \cdots A^{n-1}B] = n, \) then given any \( T > 0 \) and any \( x_0, x_1 \in \mathbb{R}^n, \) there exists a continuous control \( u \) which transfers \( x \) from \( x_0 \) at \( t = 0 \) to \( x_1 \) at \( t = T. \) (Hint: consider a change of time scale as follows: \( \forall t \geq 0, \) let \( h(t) = \int_0^t g(\sigma) d\sigma; \) show that \( h^{-1} \) is defined and \( C^1 \) on \( \mathbb{R}^+, \) and define, \( \forall t \in \mathbb{R}^+, z(t) = x(h^{-1}(t)); \) compute \( \frac{d}{dt} z(t) \) and note the simplification obtained ...).

5. Let \( A \) and \( F \) be constant \( n \times n \) matrices. Let \( b \in \mathbb{R}^n \) and let \( G = A - F. \) Show that it is possible to drive the state \( x \) of the system

\[
\dot{x}(t) = e^{Ft}Ae^{-Ft}x(t) + e^{Ft}bu(t)
\]

from any state at \( t = 0 \) to \( 0 \) at \( t = 1 \) using a continuous control \( u : [0, 1] \to \mathbb{R} \) if and only if \( \det[b \ Gb \ G^2b \cdots G^{n-1}b] \neq 0. \)