1. Let \( t_0, t_1 \in \mathbb{R} \) with \( t_0 < t_1 \). Recall that \( \mathcal{F}([t_0, t_1]; \mathbb{R}^m) \) (equipped with the usual operations of additions of functions and multiplication by a scalar) denotes the \( \mathbb{R} \)–vector space of \( \mathbb{R}^m \)–valued functions on \([t_0, t_1]\), and that \( \mathcal{C}_p^0([t_0, t_1]; \mathbb{R}^m) \) denotes the set of \( \mathbb{R}^m \)–valued piecewise-continuous functions on \([t_0, t_1]\) (as defined in class).

(i) Show that \( \mathcal{C}_p^0([t_0, t_1]; \mathbb{R}^m) \) is a vector subspace of the \( \mathbb{R} \)–vector space \( \mathcal{F}([t_0, t_1]; \mathbb{R}^m) \) (and hence is itself a \( \mathbb{R} \)–vector space, with the vector space operations inherited from \( \mathcal{F}([t_0, t_1]; \mathbb{R}^m) \)).

(ii) Show that the \( \mathbb{R} \)–vector space \( \mathcal{C}_p^0([t_0, t_1]; \mathbb{R}^m) \) is infinite-dimensional.

2. In this problem, we will prove a linear algebra result that we used in our study of controllability for linear time-varying systems.

(a) Let \( V \) be a real vector space with inner product \( \langle \cdot, \cdot \rangle \). Let \( L : V \to V \) be a (continuous) linear mapping, and assume it is symmetric, i.e. \( \forall u, v \in V: \langle Lu, v \rangle = \langle u, Lv \rangle \). Show that we then have \( \text{Im}(L) \perp \text{Ker}(L) \), where \( \perp \) denotes orthogonality with respect to the inner product \( \langle \cdot, \cdot \rangle \).

(b) Continuing (a), show that if \( V \) is additionally finite-dimensional, then \( V = \text{Im}(L) \oplus \text{Ker}(L) \).

(c) Construct a real \( 2 \times 2 \) matrix \( M \) and a vector \( x_1 \in \mathbb{R}^2 \) such that \( x_1 \notin \text{Im}(M) \) and such that \( \forall x_2 \in \text{Ker}(L) \) we have \( x_2^T x_1 = 0 \). (Note that \( M \) must necessarily be non-symmetric. This shows that in our study of controllability, symmetry of the controllability gramian \( W(t_0, t_1) \) played an essential role.)

3. Consider the Linear Time Varying system given by

\[
\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad t \in J,
\]

where \( J \subset \mathbb{R} \) is some interval of \( \mathbb{R} \), and \( A, B \) are continuous on \( J \). Recall the controllability Gramian \( W \) of this system, defined for the pair \((t_0, t_1)\) (where \( t_0, t_1 \in J, \ t_1 > t_0 \)) by:

\[
W(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_0, \tau)B(\tau)B^T(\tau)\Phi^T(t_0, \tau)d\tau.
\]

Using the algebraic matrix equation satisfied by the controllability Gramian, show that if for some \( t_0, t_1 \in J \) with \( t_0 < t_1 \) and for each \( x_0, x_1 \in \mathbb{R}^n \), there exists a continuous control on \([t_0, t_1]\) that steers the system from \( x_0 \) at \( t_0 \) to \( x_1 \) at \( t_1 \), then for each \( t_2 \in J \) with \( t_2 > t_1 \) and for each \( x_0, x_2 \in \mathbb{R}^n \), there exists a continuous control on \([t_0, t_2]\) that steers the system from \( x_0 \) at \( t_0 \) to \( x_2 \) at \( t_2 \).
4. Consider the linear time-varying control system given by $\dot{x}(t) = g(t) (Ax(t) + Bu(t))$, $t \in \mathbb{R}$, where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ for all $t \in \mathbb{R}$. $A$ is an $n \times n$ real matrix, and $B$ an $n \times m$ real matrix. Assume also that $g$ is continuous on $\mathbb{R}$ and that there exist $\alpha, \beta > 0$ such that $\forall t \in \mathbb{R}$, $0 < \alpha \leq g(t) \leq \beta < \infty$.

Show that if $\text{rank}[B \ AB \ \cdots \ A^{n-1}B] = n$, then given any $T > 0$ and any $x_0, x_1 \in \mathbb{R}^n$, there exists a continuous control $u$ which transfers $x$ from $x_0$ at $t = 0$ to $x_1$ at $t = T$. (Hint: consider a change of time scale as follows: $\forall t \geq 0$, let $h(t) = \int_0^t g(\sigma) d\sigma$; show that $h^{-1}$ is defined and $C^1$ on $\mathbb{R}^+$, and define, $\forall t \in \mathbb{R}^+$, $z(t) = x(h^{-1}(t))$; compute $\frac{d}{dt}z(t)$ and note the simplification obtained ...).

5. Let $A$ and $F$ be constant $n \times n$ matrices. Let $b \in \mathbb{R}^n$ and let $G = A - F$.

Show that it is possible to drive the state $x$ of the system $\dot{x}(t) = e^{Ft}Ae^{-Ft}x(t) + e^{Ft}bu(t)$ from any state at $t = 0$ to $0$ at $t = 1$ using a continuous control $u : [0, 1] \to \mathbb{R}$ if and only if $\text{det}[b \ Gb \ G^2b \ \cdots \ G^{n-1}b] \neq 0.$