1. (10 pts) Consider the linear time-invariant system $\dot{x}(t) = Ax(t) + bu(t)$ with $x(0) = 0$. Here $x(t), b \in \mathbb{R}^n$ and $A \in M_n(\mathbb{R})$. Assume the control $u \in C^0_p([0,1];\mathbb{R})$ is such that $u(t + 1/3) = u(t)$ for all $t \in \mathbb{R}$. Show that it is possible to drive $x$ from $x(0) = 0$ at $t = 0$ to $x(1) = x_1$ at $t = 1$ if and only if there exists $\eta \in \mathbb{R}^n$ such that

$$(I + e^{A/3} + e^{2A/3}) \begin{bmatrix} b & A b & A^2b & \cdots & A^{n-1}b \end{bmatrix} \eta = x_1.$$ 

2. (10 pts) In many concrete engineering applications of control systems, the output is observed not on a time interval, but only at certain time instants. This problem studies exactly such a situation, namely observability from “sampled observations”.

Consider the linear time-invariant system

$$\dot{x}(t) = Ax(t)$$
$$y(t) = Cx(t)$$

where $x(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}^p$ for all $t \in \mathbb{R}$. Recall that as a consequence of the Cayley-Hamilton theorem, there exist analytic $(A -$dependent) functions $\alpha_i : \mathbb{R} \rightarrow \mathbb{R}, i = 0, \ldots, n - 1$, such that $e^{At} = \sum_{i=0}^{n-1} \alpha_i(t)A^i$ for all $t \in \mathbb{R}$.

Let $T > 0$ and assume there exist $0 \leq t_0 < t_1 < \cdots < t_{n-1} \leq T$ such that the family $\{e^{A t_k}\}_{k=0}^{n-1}$ is a linearly independent family of $M_n(\mathbb{R})$.

(a) Show that under the above assumption $\{A^i\}_{i=0}^{n-1}$ is a linearly independent family in $M_n(\mathbb{R})$. Deduce from this that the determinant of the matrix with $(i,j)$ entry given by $\alpha_i(t_j)$ is non-zero.

(b) Using the above, show that if $(C,A)$ is an observable pair, then the initial condition $x(0)$ can be determined uniquely from $y(t_0), y(t_1), \cdots, y(t_{n-1})$ (i.e. a finite number of observations of the output - at appropriate time instants - suffice to recover the initial condition). (Hint: Rephrase the problem in terms of the injectivity of a certain linear operator, as was done when deriving conditions for observability).

3. (10 pts) Consider the linear time-invariant control system

$$\begin{cases}
\dot{x}(t) = Ax(t) + Bu(t) \\
y(t) = Cx(t) + Du(t)
\end{cases}$$

where $x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, y(t) \in \mathbb{R}^p$ for all $t \in \mathbb{R}$. Let $T$ be a non-singular $n \times n$ matrix, and let $P$ be a non-singular $p \times p$ matrix. Define $\tilde{x}$ by $\tilde{x}(t) = Tx(t) \forall t \in \mathbb{R}$, and $\tilde{y}$ by $\tilde{y}(t) = Py(t) \forall t \in \mathbb{R}$.
(a) Show that $\tilde{x}$ and $\tilde{y}$ satisfy

$$
\begin{align*}
(II) \left\{ \begin{array}{l}
\dot{\tilde{x}}(t) = \mathbf{T}\mathbf{A}^{-1}\tilde{x}(t) + \mathbf{T}\mathbf{B}\mathbf{u}(t) \\
\tilde{y}(t) = \mathbf{P}\mathbf{C}\mathbf{T}^{-1}\tilde{x}(t) + \mathbf{P}\mathbf{D}\mathbf{u}(t)
\end{array} \right.
\end{align*}
$$

(b) Show that system (I) is controllable iff system (II) is controllable.

(c) Show that system (I) is observable iff system (II) is observable.

(d) Let $\mathcal{O}(\mathbf{C},\mathbf{A}) = [\mathbf{C}^T \mathbf{A}^T \mathbf{C}^T \cdots (\mathbf{A}^{-1})^T \mathbf{C}^T]^T$ be the observability matrix of system (I). Show that $\text{Ker}(\mathcal{O}(\mathbf{C},\mathbf{A}))$ is an $\mathbf{A}$-invariant subspace of $\mathbb{R}^n$.

4. (5 pts) Show that the linear time-invariant control system $\dot{x}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$ is controllable iff the system

$$
\begin{align*}
\left\{ \begin{array}{l}
\dot{x}(t) = \mathbf{A}^T\mathbf{x}(t) \\
y(t) = \mathbf{B}^T\mathbf{x}(t)
\end{array} \right.
\end{align*}
$$

is observable.