1. We have, with \( x(t_0) = z(t_0) = 0 \):

\[
y(t) = (\sin t)C x(t) + (\cos t) C z(t) \\
= (\sin t) \int_{t_0}^t e^{A(t-\sigma)} (\sin \sigma) B u(\sigma) d\sigma + (\cos t) \int_{t_0}^t e^{A(t-\sigma)} (\cos \sigma) B u(\sigma) d\sigma \\
= \int_{t_0}^t e^{A(t-\sigma)} (\sin t \sin \sigma + \cos t \cos \sigma) B u(\sigma) d\sigma \\
= \int_{t_0}^t e^{A(t-\sigma)} \cos(t-\sigma) B u(\sigma) d\sigma \\
= \int_{t_0}^t T(t, \sigma) u(\sigma) d\sigma,
\]
with \( T(t, \sigma) = e^{A(t-\sigma)} \cos(t-\sigma) B = \tilde{T}(t, \sigma) \), where \( \tilde{T}(\tau) = e^{A\tau} \cos(\tau) B \) for all \( \tau \).

2. \( \text{Ker}(\mathcal{O}_{C,A}) \cap \text{Im}(\mathcal{C}_{A,B}) \) is a vector subspace of \( \mathbb{R}^n \); assume \( \dim(\text{Ker}(\mathcal{O}_{C,A}) \cap \text{Im}(\mathcal{C}_{A,B})) = l \). Assume also \( \dim(\text{Ker}(\mathcal{O}_{C,A}) = p \) and \( \dim(\text{Im}(\mathcal{C}_{A,B})) = q \). Let \( \{u_1, \cdots, u_l, v_1, \cdots, v_{q-l}, w_{p-l}, \cdots, w_{n} \} \) be \( n \) linearly independent vectors of \( \mathbb{R}^n \) such that:

- \( \{u_1, \cdots, u_l\} \) span \( \text{Ker}(\mathcal{O}_{C,A}) \cap \text{Im}(\mathcal{C}_{A,B}) \),
- \( \{u_1, \cdots, u_l, v_1, \cdots, v_{q-l}\} \) span \( \text{Im}(\mathcal{C}_{A,B}) \),
- \( \{u_1, \cdots, u_l, w_{p-l}, \cdots, w_{p}\} \) span \( \text{Ker}(\mathcal{O}_{C,A}) \).

Let \( \{e_1, e_2, \cdots, e_n\} \) be the canonical basis of \( \mathbb{R}^n \) and let \( T: \mathbb{R}^n \to \mathbb{R}^n \) be the isomorphism defined by:

- \( T(e_i) = v_i \) for \( i = 1, \cdots, q-l \),
- \( T(e_i) = u_i \) for \( i = q-l+1, \cdots, q \),
- \( T(e_i) = z_{i+p-l} \) for \( i = q+1, \cdots, n+l-p \),
- \( T(e_i) = w_{i-n-l+p} \) for \( i = n+l-p+1, \cdots, n \).

Then, for \( i = 1, \cdots, q-l, \) \( T^{-1} A T(e_i) = T A (v_i) \); note now that the \( v_j \) lie in the \( A \)-invariant subspace \( \text{Im}(\mathcal{C}_{A,B}) \) (though they do not span that whole subspace). As a result, \( T A (v_i) \) lies in that same subspace, and since \( \text{Im}(\mathcal{C}_{A,B}) \) is spanned by the \( u_j \) and the \( v_k \), \( T A (v_i) \) can be written as a linear combination of the \( u_j \) and the \( v_k \). Similarly, for \( i = q-l+1, \cdots, q, \) \( T^{-1} A T(e_i) = T A (u_i) \), but since the \( u_j \) lie in the \( A \)-invariant subspace \( \text{Ker}(\mathcal{O}_{C,A}) \cap \text{Im}(\mathcal{C}_{A,B}) \), \( T A (u_i) \) lies again in \( \text{Ker}(\mathcal{O}_{C,A}) \cap \text{Im}(\mathcal{C}_{A,B}) \). Furthermore, since the \( u_j \) span this whole subspace, \( T A (u_i) \) can be written as a linear combination of the \( u_j \). Now, for \( i = q+1, \cdots, n+l-p, T^{-1} A T(e_i) = T A (z_{i+p-l}) \), which can be written as a linear combination of the basis vectors \( u_j, v_k, w_l, z_r \) of \( \mathbb{R}^n \). Finally, for \( i = n+l-p+1, \cdots, n \), \( T^{-1} A T(e_i) = T A (w_{i-n-l+p}) \), and by a similar reasoning as above, it is easy to see that \( T A (w_{i-n-l+p}) \) can be written as a linear combination of the \( u_j \) and the \( w_k \). It directly follows from all this discussion that the matrix of \( T^{-1} A T \) (in the canonical basis) has exactly the form indicated. Furthermore, since the column vectors of \( B \) lie in the subspace \( \text{Im}(\mathcal{C}_{A,B}) \), the column vectors of \( T^{-1} B \) will lie in the span of the \( e_i \), for \( i = 1, \cdots, q \). This shows that \( T^{-1} B \) must have the structure indicated. Finally, since the \( u_j \) and the \( w_k \) span \( \text{Ker}(\mathcal{O}_{C,A}) \), we have \( C(u_j) = C(w_k) = 0 \) (for all indices \( j, k \) in their range), and this means \( CT(e_i) = 0 \) for \( i = q-l+1, \cdots, q \) and \( i = n+l-p+1, \cdots, n \). As a result, \( CT \) has exactly the structure indicated.

Let us now show that the pair \( \left( \begin{pmatrix} A_1 & 0 \\ A_2 & A_3 \end{pmatrix}, \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \right) \) is controllable. Well, the controllability matrix of the pair \( (A,B) \) has rank \( q \) (the dimension of the subspace \( \text{Im}(\mathcal{C}_{A,B}) \)). Since \( T \) is a non-singular transformation, the controllability matrix of the pair \( (T^{-1} A T, T^{-1} B) \) also has rank \( q \). Now if we write down the controllability matrix
of this latter pair, we end up with the controllability matrix of the pair \( \left( \begin{pmatrix} A_1 & 0 \\ A_2 & A_3 \end{pmatrix}, \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \right) \) on the first \( q \) rows, followed by \( n - q \) identically zero rows. For that matrix to have rank \( q \), the controllability matrix of the pair \( \left( \begin{pmatrix} A_1 & 0 \\ A_2 & A_3 \end{pmatrix}, \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \right) \) must have full rank (i.e. \( q \)), that is, this pair must be controllable. Let us now show that this implies that the pair \((A_1, B_1)\) is controllable as well. Here again, the controllability matrix of the pair \( \left( \begin{pmatrix} A_1 & 0 \\ A_2 & A_3 \end{pmatrix}, \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \right) \) contains the controllability matrix of the pair \((A_1, B_1)\) on its first rows. For the controllability matrix of the pair \( \left( \begin{pmatrix} A_1 & 0 \\ A_2 & A_3 \end{pmatrix}, \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \right) \) to be full rank, that of the pair \((A_1, B_1)\) must be full rank as well. Hence the conclusion. The result for the observable pairs is obtained through identical reasoning.

3. Assume \( Q \) is not identically zero on \( \mathbb{R}^+ \); we shall show that there exists then \( u : \mathbb{R}^+ \to \mathbb{R}^m \) continuous and \( T > 0 \) such that \( \int_0^T Q(\tau)u(\tau)d\tau \neq 0 \). If \( Q \) is not identically zero on \( \mathbb{R}^+ \) then there exist indices \( 1 \leq i \leq n, 1 \leq j \leq m, \) and \( t \geq 0 \), such that \( Q_{i,j}(t) \neq 0 \). Since \( Q \) is assumed continuous (i.e. all its entries are continuous), \( Q_{i,j} \) is a continuous function on \( \mathbb{R}^+ \). Hence, there exists \( \epsilon > 0 \) such that \( Q_{i,j}(t+\delta) \neq 0 \) for all \( 0 < \delta < \epsilon \). Let \( T = t + \epsilon \), and define \( u : \mathbb{R}^+ \to \mathbb{R}^m \) such that \( u_k(t) = 0 \) for all \( k \neq j \) and all \( t \geq 0 \), and such that \( u_j(t) = Q_{i,j}(t) \) \( \forall t \geq 0 \). \( u \) is clearly continuous on \( \mathbb{R}^+ \). The \( i^\text{th} \) row of \( \int_0^T Q(\tau)u(\tau)d\tau \) is then given by \( \int_0^T (Q_{i,j}(\tau))^2d\tau \), and since

\[
\int_0^T (Q_{i,j}(\tau))^2d\tau \geq \int_t^{t+\epsilon} (Q_{i,j}(\tau))^2d\tau > 0,
\]

we have that \( \int_0^T Q(\tau)u(\tau)d\tau \neq 0 \). This proves the result.

4. Assume \((A, B)\) is a non-controllable pair, and let \( r = \text{rank}(C_{(A, B)}) \), where \( C_{(A, B)} \) is the controllability matrix of the pair \((A, B)\). By the Controllability Normal Form theorem, there exists \( T \in M_n(\mathbb{R}) \), \( T \) non-singular, such that:

\[
TAT^{-1} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{pmatrix}, \quad TB = \begin{pmatrix} \tilde{B}_1 \\ 0 \end{pmatrix},
\]

where \( \tilde{A}_{11} \in M_r(\mathbb{R}), \tilde{B}_1 \in M_{r,m}(\mathbb{R}) \), and the pair \((\tilde{A}_{11}, \tilde{B}_1)\) is controllable. Hence, by non-singularity of \( T \), we have, \( \forall \lambda \in \mathbb{C} \):

\[
\text{rank}([A - \lambda I \ B]) = \text{rank}(T([A - \lambda I \ B]) T) = \text{rank}([T A T^{-1} - \lambda I \ T \ B]) = \text{rank}(\begin{pmatrix} \tilde{A}_{11} - \lambda I & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} - \lambda I \end{pmatrix} T \begin{pmatrix} \tilde{B}_1 \\ 0 \end{pmatrix}) = \text{rank}(\begin{pmatrix} \tilde{A}_{11} - \lambda I & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} - \lambda I \end{pmatrix} \begin{pmatrix} \tilde{B}_1 \\ 0 \end{pmatrix}) \leq \text{rank}(\tilde{A}_{11} - \lambda I \tilde{A}_{12} \tilde{B}_1) + \text{rank}(\tilde{A}_{22} - \lambda I) \leq r + \text{rank}(\tilde{A}_{22} - \lambda I);
\]

Let now \( \lambda \in \mathbb{C} \) be an eigenvalue of \( \tilde{A}_{22} \). Then, with that choice of \( \lambda \), we obtain \( \text{rank}(\tilde{A}_{22} - \lambda I) < n - r \), and, as a result:

\[
\text{rank}([A - \lambda I \ B]) < r + (n - r) = n,
\]

which proves the desired result.

5. \( A \) has distinct eigenvalues \( \Rightarrow A \) is diagonalizable. There then exists \( T \in M_n(\mathbb{C}) \) (note that \( T \) need not have real entries) such that \( TAT^{-1} = \Lambda \), where \( \Lambda \in M_n(\mathbb{C}) \) is diagonal. Since \( A \) is non-singular, so is \( \Lambda \), and since the latter is diagonal, none of its diagonal entries (which are its eigenvalues) are 0. Let \( U \in M_n(\mathbb{C}) \) be the diagonal matrix with \((i, i)\) entry given by \( \log(A_{ii}) \) (with any determination of \( \log \) that one desires). Then \( e^U = \Lambda \), as is easily verified, and hence, letting \( C = T^{-1}UT \), we obtain \( e^C = e^{T^{-1}UT} = T^{-1}e^UT = T^{-1}A \Lambda T = A \), as desired.
6. Let $\mathbf{A} \in M_n(\mathbb{R})$, and let $r = \text{rank}(\mathbf{A})$. This means there exists $r$ column vectors of $\mathbf{A}$ which form a linearly independent family (and there is no larger such family). This implies in particular that a $r \times r$ submatrix of $\mathbf{A}$ consisting of those column vectors and $r$ rows of $\mathbf{A}$ is non-singular. This implies therefore that $\text{rank}(\mathbf{A}) \leq \text{rank}(\mathbf{A}^T)$. Repeating the same inequality with $\mathbf{A}^T$ yields the reverse inequality, and hence $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}^T)$. 