1. (a) 

\[ \dot{x}(t) = \begin{pmatrix} -5 & 3 \\ 3 & -5 \end{pmatrix} x(t) + \begin{pmatrix} 2 \\ 2 \end{pmatrix} u(t), \quad t \geq 0. \]

The characteristic polynomial of the matrix \( A = \begin{pmatrix} -5 & 3 \\ 3 & -5 \end{pmatrix} \) is given by \((s + 5)^2 - 9\), and its roots, i.e. the eigenvalues of \( A \), are \( \lambda_1 = -2, \lambda_2 = -8 \), which are both in \( \mathbb{C}^- \). Hence \( A \) is Hurwitz, and a stabilizing feedback law is given by \( u = Fx \), with \( F = 0 \).

(b) 

\[ \dot{x}(t) = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} u(t), \quad t \geq 0. \]

The characteristic polynomial of the matrix \( A = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} \) is given by \((s - 5)^2 - 9\), and hence the eigenvalues of \( A \) are given by \( \lambda_1 = 2, \lambda_2 = 8 \), none of which is in \( \mathbb{C}^- \). Hence both modes of \( A \) are unstable. The controllability matrix of the pair \( (A, b) \) (with \( b = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \)) is given by \( C_{(A,b)} = \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix} \), which has rank 1, and since this rank is strictly less than 2, the pair \( (A, b) \) is not controllable. It follows that there cannot exist any stabilizing linear static feedback law for this system since the uncontrollable mode is unstable.

(c) 

\[ \dot{x}(t) = \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix} x(t) + \begin{pmatrix} 3 \\ 3 \end{pmatrix} u(t), \quad t \geq 0. \]

The characteristic polynomial of the matrix \( A = \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix} \) is given by \((s - 2)^2 - 16\), and hence the eigenvalues of \( A \) are given by \( \lambda_1 = -2, \lambda_2 = 6 \). Hence \( A \) has one stable mode (corresponding to \( \lambda_1 \)) and one unstable mode (corresponding to \( \lambda_2 \)); if this unstable mode is controllable, then the pair \( (A, b) \) (with \( b = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \)) is stabilizable and a stabilizing linear static state feedback law does exist. Otherwise, no such stabilizing feedback law can exist. To determine whether the unstable mode is controllable, we have to transform the pair \( (A, b) \) into the standard controllability form. For this, we have to refer to the proof of the standard controllability form theorem in order to construct a suitable isomorphism of \( \mathbb{R}^n \) (i.e. change of basis). More specifically, we have to construct a basis of \( \mathbb{R}^n \) adapted to the controllability subspace of the pair \( (A, b) \), that is, to the image of the controllability matrix \( C_{(A,b)} \) of the pair \( (A, b) \). We have: \( C_{(A,b)} = \begin{pmatrix} 3 & 18 \\ 3 & 18 \end{pmatrix} = \langle b \rangle \langle 6b \rangle \), from which we deduce that \( Im(C_{(A,b)}) = \langle b \rangle \) (denoting linear span). Let then \( f_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) and let \( f_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \). Clearly \( f_1 \in Im(C_{(A,b)}) \) and \( (f_1, f_2) \) is a basis of \( \mathbb{R}^2 \). Let now \( P : \mathbb{R}^2 \to \mathbb{R}^2 \) be the isomorphism defined by \( Pf_i = e_i, i = 1, 2 \), where \( (e_1, e_2) \) is the canonical basis of \( \mathbb{R}^2 \). Identifying \( P \) with its matrix representation in the canonical basis of \( \mathbb{R}^2 \), we can write:

\[ P = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}. \]

We obtain therefore:

\[ PAP^{-1} = \begin{pmatrix} 6 & 4 \\ 0 & -2 \end{pmatrix}, \quad Pb = \begin{pmatrix} 3 \\ 0 \end{pmatrix}. \]

Note that the “\( A_{22} \)” block of \( A \), which corresponds to the uncontrollable modes, is (the one by one matrix) \(-2\), which is stable (since \(-2 \in \mathbb{C}^- \)). Hence the uncontrollable modes of this system being stable, the system is...
We have:
\[
\mathbf{PAP}^{-1} + \mathbf{PbFP}^{-1} = \begin{pmatrix}
6 & 4 \\
0 & -2
\end{pmatrix} + \begin{pmatrix}
3a & 3b \\
0 & 0
\end{pmatrix} = \begin{pmatrix}
3a + 6 & 3b + 4 \\
0 & -2
\end{pmatrix}.
\]

The matrix \( \mathbf{PAP}^{-1} + \mathbf{PbFP}^{-1} \) can be made Hurwitz by choosing \( a = -3, b = 0 \) (this is by no means the only choice, but it is certainly a simple one). For this choice, we have \( \mathbf{F} = \begin{pmatrix}
-3 & 0
\end{pmatrix} \mathbf{P} = \begin{pmatrix}
-3 & 0
\end{pmatrix} \). Hence, for this choice of \( \mathbf{F} \), the matrix \( \mathbf{PAP}^{-1} + \mathbf{PbFP}^{-1} = \mathbf{P}(\mathbf{A} + \mathbf{bF})\mathbf{P}^{-1} \) is Hurwitz, which implies that \( \mathbf{A} + \mathbf{bF} \) is Hurwitz (recall that if \( \mathbf{M} \) is any real or complex \( n \times n \) matrix and \( \mathbf{T} \) is any non-singular real or complex \( n \times n \) matrix, then \( \mathbf{M} \) and \( \mathbf{T}\mathbf{MT}^{-1} \) have the same characteristic polynomial and hence the same eigenvalues).

(d)
\[
\dot{x}(t) = \begin{pmatrix}
3 & 6 \\
6 & 3
\end{pmatrix} x(t) + \begin{pmatrix}
1 \\
-1
\end{pmatrix} u(t), \quad t \geq 0.
\]

The characteristic polynomial of the matrix \( \mathbf{A} = \begin{pmatrix}
3 & 6 \\
6 & 3
\end{pmatrix} \) is given by \((s - 3)^2 - 36\), and hence the eigenvalues of \( \mathbf{A} \) are given by \( \lambda_1 = -3, \lambda_2 = 9 \). Hence \( \mathbf{A} \) has one stable mode (corresponding to \( \lambda_1 \)) and one unstable mode (corresponding to \( \lambda_2 \)); if this unstable mode is controllable, then the pair \( (\mathbf{A}, \mathbf{b}) \) (with \( \mathbf{b} = \begin{pmatrix}
1 \\
1
\end{pmatrix} \)) is stabilizable and a stabilizing linear static state feedback law does exist. Otherwise, no such stabilizing feedback law can exist. To determine whether the unstable mode is controllable, we again have to transform the pair \( (\mathbf{A}, \mathbf{b}) \) into the standard controllability form. More specifically, we have to construct a basis of \( \mathbb{R}^n \) adapted to the controllability subspace of the pair \((\mathbf{A}, \mathbf{b})\), that is, to the image of the controllability matrix \( \mathcal{C}(\mathbf{A}, \mathbf{b}) \) of the pair \( (\mathbf{A}, \mathbf{b}) \). We have:
\[
\mathcal{C}(\mathbf{A}, \mathbf{b}) = \begin{pmatrix}
1 & -3 \\
-1 & 3
\end{pmatrix} = (\mathbf{b} - 3\mathbf{b}), \text{ from which we deduce that } \text{Im}(\mathcal{C}(\mathbf{A}, \mathbf{b})) = (\mathbf{b}). \text{ Let then } \mathbf{f}_1 = \begin{pmatrix}
1 \\
-1
\end{pmatrix} \text{ and let }
\mathbf{f}_2 = \begin{pmatrix}
0 \\
1
\end{pmatrix}. \text{ Clearly } \mathbf{f}_1 \in \text{Im}(\mathcal{C}(\mathbf{A}, \mathbf{b})) \text{ and } (\mathbf{f}_1, \mathbf{f}_2) \text{ is a basis of } \mathbb{R}^2. \text{ Let now } \mathbf{P} : \mathbb{R}^2 \to \mathbb{R}^2 \text{ be the isomorphism defined by } \mathbf{P}\mathbf{f}_i = \mathbf{e}_i, \ i = 1, 2, \text{ where } (\mathbf{e}_1, \mathbf{e}_2) \text{ is the canonical basis of } \mathbb{R}^2. \text{ Identifying } \mathbf{P} \text{ with its matrix representation in the canonical basis of } \mathbb{R}^2, \text{ we can write: }
\[
\mathbf{P} = \begin{pmatrix}
1 & 0 \\
-1 & 1
\end{pmatrix}^{-1} = \begin{pmatrix}
1 & 0 \\
1 & 1
\end{pmatrix}.
\]

We obtain therefore:
\[
\mathbf{PAP}^{-1} = \begin{pmatrix}
-3 & 6 \\
0 & 9
\end{pmatrix}.
\]

Note that the “\( \mathbf{A}_{22} \)” block of \( \mathbf{A} \), which corresponds to the uncontrollable modes, is (the one by one matrix) \( 9 \), which is unstable (since \( 9 \notin \mathbb{C}^- \)). Hence the uncontrollable modes of this system being unstable, no stabilizing linear state feedback law does exist.

2. We have:
\[
\begin{pmatrix}
\mathbf{A}_{11} & \mathbf{A}_{12} \\
0 & \mathbf{A}_{22}
\end{pmatrix} = \mathbf{T}\mathbf{U}^{-1} \begin{pmatrix}
\mathbf{A}_{11} & \mathbf{A}_{12} \\
0 & \mathbf{A}_{22}
\end{pmatrix} \mathbf{U}^{-1},
\]
and
\[
\begin{pmatrix}
\mathbf{B}_1 \\
0
\end{pmatrix} = \mathbf{T}\mathbf{U}^{-1} \begin{pmatrix}
\mathbf{B}_1 \\
0
\end{pmatrix}.
\]

Hence, \( \begin{pmatrix}
\mathcal{C}(\mathbf{A}_{11}, \mathbf{tB}_1) \\
0
\end{pmatrix} \) = \( \mathbf{T}\mathbf{U}^{-1} \begin{pmatrix}
\mathcal{C}(\mathbf{A}_{11}, \mathbf{B}_1) \\
0
\end{pmatrix} \) (where the bottom zero indicates a zero block), and since the controllability matrices of \( (\mathbf{A}_{11}, \mathbf{B}_1) \) and \( (\mathbf{A}_{11}, \mathbf{B}_1) \) are both of rank \( r \), it follows that \( \mathbf{T}\mathbf{U}^{-1} \) has the form:
\[
\mathbf{T}\mathbf{U}^{-1} = \begin{pmatrix}
\mathbf{C}_{11} & \mathbf{C}_{12} \\
0 & \mathbf{C}_{22}
\end{pmatrix}.
\]
where $C_{11} \in M_r(\mathbb{R})$. Since $TU^{-1}$ is non-singular, it follows that $C_{11}$ and $C_{22}$ are both non-singular. The inverse $UT^{-1}$ of $TU^{-1}$ has the form

$$UT^{-1} = \begin{pmatrix} C^{-1}_{11} & D_{12} \\ 0 & C^{-1}_{22} \end{pmatrix},$$

for some matrix $D_{12}$, and hence, we obtain (from the first two relations above) that

$$\hat{A}_{11} = C_{11} \hat{A}_{11} C_{11}^{-1}, \quad \hat{A}_{22} = C_{22} \hat{A}_{22} C_{22}^{-1};$$

in particular, $\hat{A}_{11}$ and $\hat{A}_{11}$ have the same spectrum. Same for $\hat{A}_{22}$ and $\hat{A}_{22}$. In other words, the definition of “controllable” and “uncontrollable” modes is independent of the particular transformation used in the standard controllability form.