Sample Midterm Problems

Problem 1

Suppose that a random process \( \{x_t, t \in \mathbb{Z}_+\} \) is given by the following dynamics:

\[
x_{t+1} = ax_t + w_t, \quad t \in \mathbb{Z}_+,\n\]

where \( \{w_t\} \) is zero-mean Gaussian and \( |a| < 1 \). Is \( \lim_{t \to \infty} E[x_t^2] < \infty \)?

Suppose now that \( |a| > 1 \). Is the process positive Harris recurrent?

Problem 2

Consider the following linear system:

\[
x_{t+1} = ax_t + p_t b u_t + w_t,
\]

where the state, control and noise realizations satisfy \( x_t \in \mathbb{R}, u_t \in \mathbb{R} \) and \( w_t \in \mathbb{R} \). Suppose \( \{w_t\} \) is i.i.d. zero-mean Gaussian with a given variance. Here \( \{p_t\} \) is a sequence of independent, identically distributed random variables which take values of 0 or 1, such that \( p_t = 1 \) with probability \( p \).

The goal is to obtain

\[
\inf_{\Pi} J(\Pi, x),
\]

where

\[
J(\Pi, x) = E_x^\Pi \left[ \sum_{t=0}^{T-1} Qx_t^2 + p_tRu_t^2 + P_Tx_T^2 \right],
\]

with \( R, Q, P_T > 0 \).

The controller has access to the information set \( I_t = \{x_s, p_s, s \leq t-1\} \cup \{x_t\} \) at time \( t \in \mathbb{Z}_+ \) and can pick an arbitrary value in \( \mathbb{R} \).

Obtain the Dynamic Programming recursion for the optimal control problem. What is an optimal control policy?

Hint: Note that when \( p = 1 \), one obtains the Riccati equation we derived in class. The Riccati equation in this case was found to be:

\[
P_t = Q + A^T P_{t+1} A - A^T P_{t+1} B ((BP_{t+1} B + R))^{-1} B^T P_{t+1} A,
\]

which in the scalar case would reduce to

\[
P_t = Q + a^2 P_{t+1} - a^2 b^2 P_{t+1}^2 / (b^2 P_{t+1} + R).
\]
Problem 3

Let a random walk on $\mathbb{Z}$ be described as follows. Let $x_t$ denote the state variable. Let the one-step transition matrix be given by $P(x_{t+1} = b - 1|x_t = b) = P(x_{t+1} = b + 1|x_t = b) = 1/2$ for all $b \in \mathbb{Z}$ and all $t \in \mathbb{N}$.

a) Suppose the initial state is $a \in \mathbb{N}$. What is $P_a\left(\min(t > 0 : x_t = 0) < \infty\right)$? Is the Markov chain Harris recurrent? Explain your answer.

b) Suppose the initial state is $0$. What is $E_0\left[\min(t > 0 : x_t = 0)\right]$? Is the Markov chain positive Harris recurrent? Explain your answer.

Problem 4

Let $X$ a random variable given by $P(|X| \leq K) = 1$ for some $K \in \mathbb{R}_+$. Let $Y_n$, $n \in \mathbb{Z}_+$, be an arbitrary sequence of random variables that are defined on a common probability space and that correlated with $X$.

Does $\lim_{n \to \infty} E[X|Y_0 = y_0, Y_1 = y_1, \ldots, Y_n = y_n]$ exist almost surely? Prove your statement.

Problem 5

Consider a Markov chain with state space $\{0, 1, 2, 3\}$ and let a one-stage probability transition matrix be given by:

$$
P = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1/2 & 0 & 1/2 \\
0 & 0 & 1/2 & 1/2 \\
\end{bmatrix},
$$

such that, for example, $P(3, 3) = 1/2$.

Compute $E[\min(t \geq 0 : x_t = 3)|x_0 = 0]$, that is the expected minimum number of stages for the state to move from 0 to 3.

Many other former problems have been converted to homework problems; therefore the homeworks will give you a good idea on what to expect in the exams.