Problem 1

Let \( \{x_t\} \) be a Markov chain defined on state space \( \{0, 1, 2\} \). Let the one-stage probability transition matrix be given by:

\[
P = \begin{bmatrix}
0 & 1 & 0 \\
1/2 & 0 & 1/2 \\
0 & 2/3 & 1/3
\end{bmatrix}
\]

Compute \( E[\min(t \geq 0 : x_t = 2)] \mid x_0 = 0 \), that is the expected minimum number of stages for the state to move from 0 to 2.

Hint: One way to solve part b is as follows: Note that if the expected minimum time to go to state 2 is from state 1 is \( t_1 \) and the expected minimum time to go to state 2, from state 0 is \( t_0 \), then the expected minimum time to go to state 2 from state 0 will be \( t_0 = 1 + P(0, 2)t_2 + P(0, 1)t_1 + P(0, 0)t_0 \), where \( t_2 = 0 \). You can follow this line of reasoning to obtain the result.

Problem 2

Let \( \{x_t, t \in \mathbb{N}\} \) be an irreducible Markov chain taking values in a countable space \( \mathbb{X} \). Let a sequence of stopping times be given by \( \tau(k) = \min(t > \tau(k-1) : x_t \in A) \) where \( A \subset \mathbb{X} \) and it is known that \( \tau(k) - \tau(k-1) < \infty \) for all \( k \) (almost surely). Let \( x_0 \in A \).

Is the sampled process at the stopping times, \( \{x_{\tau(k)}, k \in \mathbb{N}\} \) a Markov Chain?

Problem 3

Consider the following two-server system:

\[
\begin{align*}
x_{t+1}^1 &= \max(x_t^1 + A_t^1 - u_t^1, 0) \\
x_{t+1}^2 &= \max(x_t^2 + A_t^2 + u_t^1 1_{\{u_t^1 \leq x_t^1 + A_t^1\}} - u_t^2, 0),
\end{align*}
\]

where \( 1_{\{\cdot\}} \) denotes the indicator function and \( A_t^1, A_t^2 \) are independent and identically distributed (i.i.d.) random variables with geometric distributions, that is, for \( i = 1, 2 \),

\[
P(A_t^i = k) = p_i(1 - p_i)^k \quad k \in \{0, 1, 2, \ldots\},
\]

for some scalars \( p_1, p_2 \) such that \( E[A_t^1] = 1.5 \) and \( E[A_t^2] = 1 \).

Suppose the control actions \( u_t^1, u_t^2 \) are such that \( u_t^1 + u_t^2 \leq 5 \) for all \( t \in \mathbb{N}_+ \) and \( u_t^1, u_t^2 \in \mathbb{R}_+ \). At any given time \( t \), the controller has to decide on \( u_t^1 \) and \( u_t^2 \) with knowing \( \{x_s^1, x_s^2, s \leq t\} \) but not knowing \( A_t^1, A_t^2 \).

Is this server system stochastically stabilizable by some policy, that is, does there exist an invariant distribution under some control policy?

If your answer is positive, provide a control policy and show that there exists a unique invariant distribution.
If your answer is negative, precisely state why.

Hint: Note that allowing the control actions to take real values, and not only integers, simplifies the problem.

Problem 4
Consider a random walk on a line of integers $\mathbb{Z}$ be described as follows. Let $x_t \in \mathbb{Z}$ denote the state variable. Let the one-step transition matrix be given by $P(x_{t+1} = b - 1 | x_t = b) = P(x_{t+1} = b + 1 | x_t = b) = 1/2$ for all $b \in \mathbb{Z}$ and all $t \in \mathbb{N}$.

a) [5 Points] Is the Markov Chain irreducible?

b) [10 Points] Suppose the initial state is 0. What is $E_0 \left[ \min(t > 0 : x_t = 0) \right]$ (recall that $E_0[]$ denotes that the initial state is 0)? Is the Markov chain positive recurrent?

c) [10 Points] Suppose the initial state is $a \in \mathbb{N}$. What is $P_a \left( \min(t > 0 : x_t = 0) < \infty \right)$? Is the Markov chain recurrent?

Hint: For part c), one strategy is as follows: First show that $\{x_t\}$ is a Martingale sequence. Define a stopping time $\tau^N = \min \left( \min(t : x_t \notin \{0, 1, 2, \ldots, K\}), N \right)$ for some $K > a$. Invoking the Martingale optional sampling theorem, deduce that $E[x_{\tau^N}] = x_0 = a$. Now take $K \to \infty$ and then take $N \to \infty$ and find out what the probability of exiting at state 0 is.

Problem 5
Consider a Controlled Markov Chain with the following dynamics:

$$x_{t+1} = ax_t + bu_t + w_t,$$

where $w_t$ is a zero-mean Gaussian noise with a finite variance, $a, b \in \mathbb{R}$ are the system dynamics coefficients. One controller policy which is admissible (that is, the policy at time $t$ is measurable with respect to $\sigma(x_0, x_1, \ldots, x_t)$ and is a mapping to $\mathbb{R}$) is the following:

$$u_t = -\frac{a + 0.5}{b}x_t.$$

Show that $\{x_t\}$, under this policy, has a unique invariant probability measure.

Problem 6
Consider a similar setup to the one in Problem 5, with $b = 1$:

$$x_{t+1} = ax_t + u_t + w_t,$$

where $w_t$ is a zero-mean Gaussian noise with a finite variance, and $a \in \mathbb{R}$ is a known number.

This time, suppose, we would like to find a control policy such that there exists an invariant probability measure $\pi$ for $\{x_t\}$ and under this invariant probability measure

$$E_\pi[u^2] < \infty.$$
Further, suppose we restrict the set of control policies to be linear, time-invariant; that is of the form $u(x_t) = kx_t$ for some $k \in \mathbb{R}$.

Find the set of all $k$ values for which there exists an invariant probability measure that has a finite second moment.

Hint: Use Foster-Lyapunov criteria.

**Problem 7**

Consider a two server-station network; where a router routes the incoming traffic, as is depicted in Figure 1.

![Figure 1](image)

Let $L^1_t, L^2_t$ denote the number of customers in stations 1 and 2 at time $t$. Let the dynamics be given by the following:

$$L^1_{t+1} = \max(L^1_t + u_t A_t - N^1_t, 0), \quad t \in \mathbb{N}.$$  
$$L^2_{t+1} = \max(L^2_t + (1 - u_t) A_t - N^2_t, 0), \quad t \in \mathbb{N}.$$  

Customers arrive according to an independent Bernoulli process, $A_t$, with mean $\lambda$. That is, $P(A_t = 1) = \lambda$ and $P(A_t = 0) = 1 - \lambda$. Here $u_t \in [0, 1]$ is the router action.

Station 1 has a Bernoulli service process $N^1_t$ with mean $n_1$, and Station 2 with $n_2$.

Suppose that a router decides to follow the following algorithm to decide on $u_t$: If a customer arrives, the router simply sends the incoming customer to the shortest queue.

Find sufficient conditions (on $\lambda, n_1, n_2$) for this algorithm to lead to a stochastically stable system with invariant measure $\pi$ which satisfies $E_\pi[L^1 + L^2] < \infty$.

**Problem 8 [Optional]**

Prove Birkhoff’s Ergodic Theorem for a countable state space; that is the result that for an irreducible Markov chain $\{x_t\}$ living in a countable space $\mathbb{X}$, which has a unique invariant distribution $\mu$, the following applies almost surely:

$$\lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} f(x_t) = \sum_{i} f(i) \mu(i),$$

for every bounded $f : \mathbb{X} \to \mathbb{R}$.

Hint: You may proceed as follows. Define a sequence of empirical occupation measures for $t \in \mathbb{Z}_+$, $A \in B(\mathbb{X})$:

$$\nu_T(A) = \frac{1}{T} \sum_{t=0}^{T-1} 1_{\{x_t \in A\}}, \quad \forall A \in \sigma(\mathbb{X}).$$
Now, define:

\[ F_t(A) = \left( \sum_{s=1}^{t} 1_{\{x_s \in A\}} - t \sum_x P(A|x)u_t(x) \right) \]

\[ = \left( \sum_{s=1}^{t} 1_{\{x_s \in A\}} - \sum_{s=0}^{t-1} \sum_x P(A|x)1_{\{x_s = x\}} \right) \quad (2) \]

Let \( F_t = \sigma(x_0, \cdots, x_t) \). Verify that, for \( t \geq 2 \),

\[ E[F_t(A) | F_{t-1}] = E\left[ \left( \sum_{s=1}^{t} 1_{\{x_s \in A\}} - \sum_{s=0}^{t-1} \sum_x P(A|x)1_{\{x_s = x\}} \right) | F_{t-1} \right] \]

\[ = E\left[ \left( 1_{\{x_t \in A\}} - \sum_x P(A|x)1_{\{x_{t-1} = x\}} \right) | F_{t-1} \right] \]

\[ + \left( \sum_{s=1}^{t-1} 1_{\{x_s \in A\}} - \sum_{s=0}^{t-2} \sum_x P(A|x)1_{\{x_s = x\}} \right) \]

\[ = 0 + \left( \sum_{s=1}^{t-1} 1_{\{x_s \in A\}} - \sum_{s=0}^{t-2} \sum_x P(A|x)1_{\{x_s = x\}} \right) | F_{t-1} \right] \quad (3) \]

\[ = F_{t-1}(A), \quad (4) \]

where the last equality follows from the fact that \( E[1_{x_t \in A}|F_{t-1}] = P(x_t \in A|F_{t-1}) \). Furthermore,

\[ |F_t(A) - F_{t-1}(A)| \leq 1. \]

Now, we have a sequence which is a martingale sequence. We will invoke a martingale convergence theorem; which is applicable for **martingales with bounded increments**. By a version of the martingale stability theorem, it follows that

\[ \lim_{t \to \infty} \frac{1}{t} F_t(A) = 0 \]

You need to now complete the remaining steps.

Hint: You can use the **Azuma-Hoeffding inequality** and the **Borel-Cantelli Lemma** to complete the steps.

**Problem 9**

One more problem may be added here.