MATH - MTHE 474/874 - Information Theory
Fall 2019

Homework # 2

Due Date: Friday October 18, 2019

Material: *Markov property, data processing, AEP and information rates.*

Readings: Sections 3.1 and 3.2 of the textbook.

The referred problems are from the textbook.

(1) Consider the binary Polya contagion process $\{Z_n\}$ described in Example 3.16 in the textbook.

(a) Determine the distributions P_{Z_1}, $P_{Z_2|Z_1}$ and $P_{Z_3|Z_2,Z_1}$ in terms of the process parameters $\rho := R/T$ and $\delta := \Delta/T$.

(b) Determine whether or not the Markov property $Z_1 \to Z_2 \to Z_3$ holds.

(2) Consider the binary finite-memory Polya contagion process $\{Z_n\}$ with memory order $M = 1$ described in Example 3.17 in the textbook.

(a) Determine the distributions P_{Z_1}, $P_{Z_2|Z_1}$ and $P_{Z_3|Z_2,Z_1}$ in terms of the process parameters $\rho := R/T$ and $\delta := \Delta/T$.

(b) Determine $H(Z_1)$, $H(Z_2|Z_1)$ and $H(Z_3|Z_2, Z_1)$ in terms of ρ and δ and compare them with each other for $\delta > 0$ and $0 < \rho < 1/2$.

(3) Let $X \to Y \to (Z, W)$ form a Markov chain; i.e., for all $(x, y, z, w) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \times \mathcal{W}$,

$$P_{X,Y,Z,W}(x, y, z, w) = P_X(x)P_{Y|X}(y|x)P_{Z,W|Y}(z, w|y).$$

Assuming that $P_{X,Y,Z,W}(x, y, z, w) > 0$ for all (x, y, z, w), show that

$$I(X; Z) + I(X; W) \le I(X; Y) + I(Z; W).$$

(4) Problem 3.3.

(5) Problem 3.5.

(6) Let $\{(X_i, Y_i)\}_{i=1}^{\infty}$ be a two-dimensional discrete memoryless source with alphabet $\mathcal{X} \times \mathcal{Y}$ and common distribution $P_{X,Y}$.

(a) Find the limit as $n \to \infty$ of the random variable

$$\frac{1}{n} \log_2 \frac{[P_{Y^n|X^n}(Y^n|X^n)]^{\alpha}}{[P_{Y^n}(Y^n)]^{1-\alpha}}$$

for a fixed parameter $0 < \alpha < 1$.

(b) Evaluate (in terms of ϵ) the limit of part (a) for $\alpha = 1/2$ and the case of $\mathcal{X} = \mathcal{Y} = \{0, 1\}$ with $P_{X,Y}$ given by $P_{X,Y}(0, 0) = P_{X,Y}(1, 1) = \frac{1-\epsilon}{2}$ and $P_{X,Y}(0, 1) = P_{X,Y}(1, 0) = \frac{\epsilon}{2}$ where $0 < \epsilon < 1/2$ is fixed.

(7) *Binary Markov Source:* Consider the binary homogeneous Markov source: $\{X_n\}_{n=1}^{\infty}$, $X_n \in \mathcal{X} = \{0, 1\}$, with

$$\Pr\{X_{n+1} = j | X_n = i\} = \begin{cases} \frac{\rho}{1+\delta} & \text{if } i = 0 \text{ and } j = 1 \\ \frac{\rho+\delta}{1+\delta} & \text{if } i = 1 \text{ and } j = 1 \end{cases},$$

where $n \geq 1$, $0 \leq \rho \leq 1$ and $\delta \geq 0$.

(a) Find the initial state distribution $(\Pr\{X_1 = 0\}, \Pr\{X_1 = 1\})$ required to make the source $\{X_n\}$ stationary.

Assume in the next questions that the source is stationary.

(b) Find the entropy rate of $\{X_n\}$ in terms of ρ and δ.

(c) If $\delta = 1$ and $\rho = 3/4$, compute the source redundancies ρ_D, ρ_M and ρ_T.

(d) Suppose that $\rho = 0$. Is $\{X_n\}$ irreducible? What is the value of the entropy rate in this case ?

(e) If $\delta = 0$, show that $\{X_n\}$ is a discrete memoryless source and compute its entropy rate in terms of ρ.

Additional Problems for MATH 874 students:

(8) Problem 3.2.

(9) Given a binary memoryless uniformly distributed process $\{X_i\}_{i=1}^{\infty}$, define the process $\{Y_i\}_{i=1}^{\infty}$ as follows

$$Y_i = X_i \oplus Z_i, \quad i = 1, 2, \ldots,$$

where $\oplus$ denotes addition modulo-2 and $\{Z_i\}_{i=1}^{\infty}$ is a binary stationary Markov process that is independent of $\{X_i\}_{i=1}^{\infty}$. Determine the mutual information rate

$$\lim_{n \to \infty} \frac{1}{n} I(X^n; Y^n).$$

(10) Consider two stationary Markov sources $\{X_i\}$ and $\{\hat{X}_i\}$ with common finite alphabet $\mathcal{X}$. Determine $\lim_{n \to \infty} \frac{1}{n} D(P_{X^n} \| P_{\hat{X}^n})$.
