Material: Transform coding and vector quantization.

(1) Let \( X_n = W_n + W_{n-1}, n = 0, 1, 2, \ldots \) where the sequence \( \{W_n\} \) is zero mean, unit variance, and uncorrelated.

(a) Find the Karhunen-Loeve transform for \( X = (X_1, X_2)^T \).

(b) Assuming that the \( W_i \)'s are Gaussian, consider the following two methods. In the first one, an equal number of bits is used to scalar quantize \( X_1 \) and \( X_2 \) separately, but optimally. In the second one, Karhunen-Loeve transform coding is used with scalar quantization and optimal bit allocation. Using high-resolution approximations, what is approximately the performance gain of the second method over the first?

(2) Let \( X \) be a two-dimensional random variable uniformly distributed over the square \([-1, 1] \times [-1, 1]\). Find two 4-point vector quantizers for \( X \) which satisfy the nearest neighbor and centroid conditions and yet yield two different MSE’s. Compute the MSE for both quantizers.

(3) Consider the distortion measure \( d(x, y) = \|A(x - y)\|^2 \), where \( x, y \in \mathbb{R}^k \) and \( A \) is a \( k \times k \) nonsingular matrix.

(a) Write out the nearest neighbor condition for a given codebook \( \{c_1, \ldots, c_N\} \subset \mathbb{R}^k \) with respect to \( d(x, y) \) and show that the resulting quantization cells are convex polytopes.

(b) Assume \( X \) is a \( k \)-dimensional random vector with finite second moment. Find the centroids for the quantization cells \( \{R_1, \ldots, R_N\} \) with respect to \( d(x, y) \). (Hint: Solve \( \nabla E[d(X, y)|X \in R_i] = 0 \) for \( y \).)
Let \( X = (X_1, \ldots, X_k) \) be a \( k \)-dimensional random vector such that each \( X_i \) has zero mean and variance \( \sigma^2 \). Let \( C = \{y_1, y_2\} \) be the codebook of a \( k \)-dimensional vector quantizer \( Q \) with smallest possible mean squared distortion among all vector quantizers with two codevectors.

(a) Show that the line connecting \( y_1 \) and \( y_2 \) passes through the origin. (Hint: use one of the Lloyd-Max conditions to express the mean of \( X \).)

(b) Derive an expression for the distortion of \( Q \) that depends only on \( \sigma^2 \), the lengths of the two codewords, and the probabilities of the two cells of \( Q \).

Additional Problems for MATH 874 students:

(5) Let \( R_0 \) denote the basic cell of a \( k \)-dimensional lattice \( \Lambda \), and assume that \( X \) is a \( k \)-dimensional random vector that is uniformly distributed over \( R_0 \).

(a) Show that if \( R_X \) denotes the autocorrelation matrix of \( X \), then the normalized second moment of \( R_0 \) is lower bounded as

\[
G(R_0) \geq \frac{1}{V(R_0)^{2/k} \det(R_X)^{1/k}}
\]

where equality holds if and only if \( R_X \) is a diagonal matrix with identical diagonal entries. (Hint: express \( G(R_0) \) in terms of the trace of \( R_X \).)

(b) Let \( A = R^{-1/2} \), that is, the \( k \times k \) symmetric and nonnegative definite matrix satisfying \( A^2 = R^{-1} \). Show that if a new basic cell \( \tilde{R}_0 \) is defined by transforming \( R_0 \) by \( A \):

\[
\tilde{R}_0 = \{Ax : x \in R_0\}
\]

then

\[
G(\tilde{R}_0) \leq G(R_0)
\]

where strict inequality holds unless \( R_X \) is a diagonal matrix with identical diagonal entries. (Hint: use part (a) and the fact that \( V(\tilde{R}_0) = |\det(A)|V(R_0) \).)

Note that random vector \( \tilde{X} = AX \) that is uniformly distributed over \( \tilde{R}_0 \) has uncorrelated components with equal variance. Thus \( A \) is a “whitening transform.” The problem
shows that an optimal lattice must have a “white” basic cell (otherwise it can always be improved).

(6) Let \( \mathbf{X} = (X_1, X_2) \) be a two-dimensional random vector whose probability density function \( f(x) \) is zero everywhere except for the four shaded sub-squares of the unit square where it has the constant value \( A \). In this problem the squared error distortion is considered. Assume that Gersho’s conjecture holds.

Let \( Q^* \) be an optimal fixed-rate two-dimensional quantizer for \( \mathbf{X} \) at some high rate \( R \) (bits per sample). Use the high-resolution formula for the distortion of the optimal fixed-rate \( k \)-dimensional vector quantizer to obtain the distortion of \( Q^* \) as a function of \( R \).