1. Let $X$ be a set, and let $(X_\alpha)_{\alpha \in \Lambda}$ be a family of subsets of $X$ with $X = \bigcup_{\alpha} X_\alpha$; assume $\forall \alpha \in \Lambda$ a topology $\tau_\alpha$ is defined on $X_\alpha$. Let $\tau$ be the set of all subsets $V \subset X$ for which $V \cap X_\alpha \in \tau_\alpha \forall \alpha$.

   (i) Show that $\tau$ is a topology on $X$.

   (ii) Give a necessary and sufficient condition for $X_\alpha$ to be open in $(X, \tau) \forall \alpha$.

   (iii) Let $M$ be a $C^\infty$ $n$–manifold, and let $\{(U_\alpha, \phi_\alpha)\}$ be a $C^\infty$ atlas on $M$; For each $\alpha$, we endowed $\bigcup_{q \in U_\alpha} T_q M$ with the unique topology which made the bijection $\Phi_q : \bigcup_{q \in U_\alpha} T_q M \to U_\alpha \times \mathbb{R}^n$ a homeomorphism, and we then endowed $TM = \bigcup_{q \in M} T_q M$ with the topology induced from the topologies in the $\bigcup_{q \in U_\alpha} T_q M$, as in (i) above. Show that in this construction, the condition obtained in (ii) is satisfied, and hence each subset $\bigcup_{q \in U_\alpha} T_q M$ is open in $TM$.

2. Let $X, Y, Z$ be $C^\infty$ manifolds; assume $X$ is a submanifold of $Y$ and $Y$ a submanifold of $Z$. Show then that $X$ is a submanifold of $Z$.

3. Let $n \in \mathbb{N}^*$, and consider $S^n$, the unit sphere of $\mathbb{R}^{n+1}$, with its usual topology (i.e. the one induced from $\mathbb{R}^{n+1}$ by inclusion). We constructed two $C^\infty$ structures on $S^n$: (i) using stereographic projections, and (ii) defining $S^n$ as the preimage of 1 under the constant rank map $x \mapsto ||x||$, which immediately established $S^n$ as a submanifold of $\mathbb{R}^{n+1} \setminus \{0\}$, and hence, of $\mathbb{R}^{n+1}$ (as follows from Problem (2) above). Prove that these two $C^\infty$ structures are identical.

4. A Lie group $G$ is a group $(G, \cdot)$ together with a $C^\infty$ manifold structure such that the group “multiplication” $m : G \times G \to G$, $(x, y) \mapsto m(x, y) = x \cdot y$ and the group inversion operation $i : G \to G$, $x \mapsto i(x) = x^{-1}$ are $C^\infty$ maps (i.e. the group structure is “compatible” with the $C^\infty$ structure).

   $\forall a \in G$, we define the maps $L_a : G \to G$, $x \mapsto L_a(x) = a \cdot x$ (“left-translation by $a$”) and $R_a : G \to G$, $x \mapsto R_a(x) = x \cdot a$ (“right-translation by $a$”).

   If $G_1, G_2$ are two Lie groups, we say a mapping $u : G_1 \to G_2$ is a Lie group homomorphism if it is both a group homomorphism and $C^\infty$.

   (a) Let $H$ be a subgroup of $G$ such that $H$ is also a submanifold of $G$. Show that $H$ is then a Lie group for the group structure and manifold structure induced by $G$ ($H$ is called a “Lie subgroup” of $G$).

   (b) Show that $\forall x, y \in G$, $T_x L_x : T_y G \to T_{xy} G$ is a vector space isomorphism.

   (c) Let $G_1, G_2$ be Lie groups. Deduce from (b) that every Lie group homomorphism $u : G_1 \to G_2$ has constant rank.

   (d) Deduce from (c) (with the same notation) that $\ker(u)$ is a Lie subgroup of $G_1$.

   (e) Show that $\forall n \geq 1$, $GL(n, \mathbb{R})$, with the usual group structure (the group operation being given by matrix multiplication) and the canonical $C^\infty$ manifold structure (as an open subset of the vector space of real $n \times n$ matrices) is a Lie group.

   (f) Show that $SL(n, \mathbb{R})$, defined as the set of all matrices in $GL(n, \mathbb{R})$ with determinant 1, is a Lie subgroup of $GL(n, \mathbb{R})$.

   (g) Let $e$ denote the identity element of $GL(n, \mathbb{R})$ (i.e. the $n \times n$ identity matrix). Show that $T_e GL(n, \mathbb{R})$ can be canonically identified with the vector space $M_n(\mathbb{R})$ of real $n \times n$ matrices.