Assignment 1, Due Sept. 27

1) Let \( \mathbb{D} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1 \} \). Show that \( \mathbb{D} \) cannot be written as a countable disjoint union of open rectangles. Show that \( \mathbb{D} \) can be written as a countable union:
\[
\mathbb{D} = \bigcup_{n \geq 1} R_n
\]
with each \( R_n \) a closed rectangle such that \( \text{int}(R_m) \cap \text{int}(R_n) = \emptyset \) for \( m \neq n \). Here \( \text{int}(R_n) \) means the interior of the rectangle \( R_n \).

2) Every real number \( x \) in the interval \( 0, 1 \] has a dyadic expansion \( x = \sum_{n=1}^{\infty} \frac{a_n}{2^n} \) with \( a_n \in \{0, 1\} \). Some numbers have two expansions: one that ends in string of 0’s and one that ends in a string of 1’s. For example \( \frac{1}{3} = 0.1000 \cdots = 0.0111 \cdots \) but this is the only ambiguity. We consider \( a_n \) to be the function on \( [0, 1] \) such that \( a_n(x) \) is the \( n^{th} \) digit of the dyadic expansion (using the non-terminating one when \( x \) has two expansions). Let \( \mathcal{A}_n \) be the \( \sigma \)-algebra generated by \( \{a_n^{-1}(O) \mid O \subset \mathbb{R} \text{ is open} \} \).

   i) Give an explicit description of the sets in \( \mathcal{A}_n \).

Let \( \mathcal{B}_n \) be \( \sigma \)-algebra generated by \( \mathcal{A}_1, \ldots, \mathcal{A}_n \).

   ii) Describe the sets in \( \mathcal{B}_2 \).

We consider \( \mathcal{B}_2 \) as the information known about \( x \) given that we know the first 2 digits.

iii) Show that \( a_3 \) is not \( \mathcal{B}_2 \)-measurable.

Let \( \mathcal{C}_n \) be the smallest \( \sigma \)-algebra containing \( \{ \mathcal{A}_n, \mathcal{A}_{n+1}, \mathcal{A}_{n+2}, \ldots \} \) and \( \mathcal{C} = \cap_{k \geq 1} \mathcal{C}_k \).

iv) Give an example of a set in \( \mathcal{C} \) (excluding \( [0, 1] \) and \( \emptyset \)).

v) Let \( \mathcal{D}_n \) be the smallest \( \sigma \)-algebra containing \( \{ \mathcal{A}_2, \mathcal{A}_3, \ldots, \mathcal{A}_n \} \). Show that \( a_1^{-1}(0) \notin \mathcal{D}_n \) for all \( n \geq 2 \). Make sure you prove your answers.

3) Let \( \mathcal{E} \) be a collection of subsets of \( X \) and \( \mathcal{A} \) the \( \sigma \)-algebra generated by \( \mathcal{E} \). For each countable subset \( \mathcal{F} \subset \mathcal{E} \) let \( \mathcal{A}_\mathcal{F} \) be the \( \sigma \)-algebra generated by \( \mathcal{F} \). Let \( \mathcal{G} = \bigcup \mathcal{A}_\mathcal{F} \) where the union runs over all countable \( \mathcal{F} \subset \mathcal{E} \). Show that \( \mathcal{A} = \mathcal{G} \).

4) Let \( B_1 \) be the \( \sigma \)-algebra of Borel subsets of \( \mathbb{R} \) and \( B_2 \) the \( \sigma \)-algebra of Borel subsets of \( \mathbb{R}^2 \). Let \( B_1 \otimes B_1 \) be the \( \sigma \)-algebra generated by sets of the form \( E \times F \) with \( E, F \in B_1 \). Show that \( B_2 = B_1 \otimes B_1 \).

5) Let \( f : [a, b] \to \mathbb{R} \) be a bounded real valued function. For each partition \( \mathcal{P} = \{t_0, t_1, \ldots, t_n\} \) with \( 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1 \), let
\[
S_\mathcal{P} = \sum_{i=1}^{n} M_i(t_i - t_{i-1})
\]
and
\[
s_\mathcal{P} = \sum_{i=1}^{n} m_i(t_i - t_{i-1})
\]
where
\[
M_i = \sup_{t_{i-1} \leq t \leq t_i} f(t) \quad \text{and} \quad m_i = \inf_{t_{i-1} \leq t \leq t_i} f(t).
\]
Furthermore we let
\[
\int_{a}^{b} f(x) \, dx = \inf_{\mathcal{P}} S_\mathcal{P}
\]
and
\[ \int_a^b f(x) \, dx = \sup_P s_P \]
where the inf and sup run over all partitions of \([0,1]\).

Let \( f(x) = \begin{cases} 
0 & \text{if } x \text{ is irrational} \\
1 & \text{if } x \text{ is rational} 
\end{cases} \). Find

\[ \int_a^b f(x) \, dx \quad \text{and} \quad \int_a^b f(x) \, dx. \]