MATH 891
Analysis I
Autumn 2017

Assignment 1, Due Sept. 27

1) In a paper from 1899 I found the equalities

$$1 - \frac{2}{5} + \frac{3}{9} - \frac{4}{13} + \cdots = \int_0^1 \frac{1}{(1+x^4)^2}\,dx$$
$$= \frac{1}{8} + \frac{3}{4}\int_0^1 \frac{1}{1+x^4}\,dx$$
$$= \frac{1}{8} + \frac{3\log(1+\sqrt{2})}{8\sqrt{2}} + \frac{3\pi}{16\sqrt{2}}$$

The author commented that he wasn't sure what the sign of the term at infinity was, so he just took the average. This seems to work numerically, but what does it all mean? This is for classroom discussion only, you don't have to hand in an answer.

2) Let $S = [0,1] \times [0,1] \subset \mathbb{R}^2$ be the unit square in the first quadrant with one corner at the origin, also let $f(x,y) = \dfrac{x^2 - y^2}{(x^2+y^2)^2}$ and $dxdy$ denote area measure in the Riemann sense. Can you evaluate $\int_S f(x,y)\,dxdy$?

3) Let Δ_0 be an equlateral triangle of area 1.

Let Δ_1 be the subset obtained by deleting the equilateral triangle whose vertices are the midpoints of the sides of Δ_0. Let Δ_2 be obtained from Δ_1 by deleting the three equilateral triangles whose vertices are midpoints of sides of the three equilateral triangles that form Δ_1. By repeating this operation we obtain a decreasing sequence of subsets: $\Delta_0 \supset \Delta_1 \supset \Delta_2 \supset \cdots$. We let $\Delta = \cap_{n\geq 0}\Delta_n$. Find the area of Δ.

4) With the Riemann integral we have the fundamental theorem of calculus as a very efficient tool for computing integrals. With the Lebesgue integral (soon to be defined) there is not such an easy tool. Thus we must establish the connection between the Riemann integral and the Lebesgue integral, i.e. when are they the same. This exercise is the first step in that direction.

Let $[a,b] \subseteq \mathbb{R}$ be a closed interval of real numbers and $f : [a,b] \to \mathbb{R}$ be a real valued function on $[a,b]$. We shall define two functions $g, h : [a,b] \to \mathbb{R}$ as follows. For $a < y < b$ we let

$$g(y) = \sup_{\delta>0} \inf_{|x-y|<\delta} f(x)$$
$$h(y) = \inf_{\delta>0} \sup_{|x-y|<\delta} f(x).$$

At the endpoints we set

$$g(a) = \sup_{\delta>0} \inf_{0\leq x-a<\delta} f(x)$$
$$h(a) = \inf_{\delta>0} \sup_{0\leq x-a<\delta} f(x)$$
$$g(b) = \sup_{\delta>0} \inf_{0\leq b-x<\delta} f(x)$$
$$h(b) = \inf_{\delta>0} \sup_{0\leq b-x<\delta} f(x).$$

Show that for all $x \in [a,b]$ we have $g(x) \leq f(x) \leq h(x)$.

5) We say that f is *lower semi-continuous at y* if $\forall\epsilon > 0\ \exists\delta > 0$ such that $f(x) > f(y) - \epsilon$ whenever $|x-y| < \delta$. We can write this loosly as "if f is larger than α at a point, then f is larger than α on a neighbourhood of a point'. In the definition above $\alpha = f(y) - \epsilon$. For example, the function $A \mapsto \text{rank}(A)$ is lower semi-continuous where A is a matrix.

We say that f is *upper semi-continuous at y* if $\forall\epsilon > 0\ \exists\delta > 0$ such that $f(x) < f(y) + \epsilon$ whenever $|x-y| < \delta$. We can write this loosly as "if f is smaller than α at a point, then f is smaller than α on a neighbourhood of a point'. In the definition above $\alpha = f(y) + \epsilon$.

i) Show that f is continuous at y if and only if it is both upper and lower semi-continuous at y.

ii) Let f be the function on $\mathbb{R}$ given by $f(x) = 0$ for $x \neq 0$ and $f(0) = 1$. What kind of continuity does f have?

6) Suppose $f : (a,b) \to \mathbb{R}$ is an real valued function on the open interval (a,b). For

$y \in (a, b)$ let[1]

$$\overline{\lim_{x \to y}} f(x) = \inf_{\delta > 0} \sup_{|x-y|<\delta} f(x)$$

and

$$\underline{\lim_{x \to y}} f(x) = \sup_{\delta > 0} \inf_{|x-y|<\delta} f(x).$$

i) Show that f is lower semi-continuous at y if and only if $f(y) \leq \underline{\lim}_{x \to y} f(x)$.

ii) Let f be as in (*ii*) in question 5. Find $\underline{\lim}_{x \to y} f(x)$ for each $y \in \mathbb{R}$.

7) Let $A \subseteq \mathbb{R}$ and $f = \chi_A$ be the indicator or characteristic function of A. Let A° be the interior of A and $\overline{A}$ the closure of A. Show that for every y

i) $\overline{\lim}_{x \to y} f(x) = \chi_{\overline{A}}(y)$

ii) $\underline{\lim}_{x \to y} f(x) = \chi_{A^\circ}(y)$

(1) Another notation for '$\overline{\lim}$' is '$\limsup$'.