Assignment 2, Due Oct. 11

1) Let \( \{ \mu_i \}_{i=1}^\infty \) be a sequence of measures on a measurable space \((X, \mathfrak{M})\) and \( \{ \alpha_i \}_{i=1}^\infty \) be a sequence with \( \alpha_i \geq 0 \). Let \( \mu(A) = \sum_{i=1}^\infty \alpha_i \mu_i(A) \). Show that \( \mu \) is a measure.

2) Let \( X \) be a set and \( \{ A_n \}_{n=1}^\infty \) be a sequence of subsets of \( X \). Let \( \chi_{A_n} \) be the indicator function of \( A_n \). Let \( \limsup_n A_n = \{ x \in X \mid x \in A_n \text{ for infinitely many } n \text{'s} \} \) and \( \liminf_n A_n = \{ x \in X \mid x \in A_n \text{ for all but finitely many } n \text{'s} \} \).

   i) Show that \( \limsup_n A_n = \bigcap_{n=1}^\infty \bigcup_{k=n}^\infty A_k \).

   ii) Show that \( \liminf_n A_n = \bigcup_{n=1}^\infty \bigcap_{k=n}^\infty A_k \).

   iii) Show that \( \limsup_n \chi_{A_n} = \chi_A \) where \( A = \limsup_n A_n \).

   iv) Show that \( \liminf_n \chi_{A_n} = \chi_B \) where \( B = \liminf_n A_n \).

3) Let \((X, \mathfrak{M})\) be a measurable space and \( \mu : X \to [0, \infty] \). We say that \( \mu \) is finitely additive if whenever \( E_1, \ldots, E_n \in \mathfrak{M} \) are pairwise disjoint we have \( \mu(\bigcup_{i=1}^n E_i) = \sum_{i=1}^n \mu(E_i) \).

If we have \( \mu(\bigcup_{i=1}^\infty E_i) \leq \sum_{i=1}^\infty \mu(E_i) \) for every sequence \( \{E_i\}_{i=1}^\infty \) (even if they are not pairwise disjoint), we say that \( \mu \) is countable subadditive.

Show that if \( \mu \) is both finitely additive and countably subadditive then it is countably additive.

In the next two exercises we make use of Lebesgue measure on the \( \sigma \)-algebra of Borel sets. For the context of these two questions we work with \( X = [0, 1] \) and \( \mathfrak{M} \) is the \( \sigma \)-algebra of Borel subsets of \([0, 1] \). We will denote Lebesgue measure by \( m \). The main point we need to know about Lebesgue measure is that if \( I \subseteq [0, 1] \) is an interval (open, closed, half-open) then \( m(I) = \ell(I) \) where \( \ell \) is the length. Thus \( m((1/2, 3/4]) = 1/4 \).

4) Let \( m \) denote Lebesgue measure on \([0, 1] \).

   i) Show that for every \( \epsilon > 0 \) there exists an open set \( G \subseteq [0, 1] \) such that \( m(G) < \epsilon \) and \( Q \cap [0, 1] \subseteq G \).

   ii) Show that there is a closed set \( C \subseteq [0, 1] \) such that \( m(C) > 0 \) but \( C \) does not contain any open interval.

5) Let \( m \) denote Lebesgue measure on \([0, 1] \), and let \( \{ A_n \}_n \) be a sequence of measurable subsets of \([0, 1] \). Suppose that there is \( \epsilon > 0 \) such that for all \( n \), \( m(A_n) > \epsilon \). Show that there is at least one \( x \in [0, 1] \) such that \( x \) is in infinitely many \( A_n \)’s.