Assignment 3, Due Nov. 8

1) Suppose that \( f \) is an essentially bounded (Lebesgue) measurable function on \([0, 1]\), and \( \|f\|_p \) be the \( p \)-norm of \( f \) with respect to Lebesgue measure. Show that
\[
\lim_{p \to \infty} \|f\|_p = \|f\|_\infty.
\]

2) For \( 0 < p < 1 \) let
\[
\|f\|_p = \left( \int_0^1 |f(x)|^p \, dx \right)^{\frac{1}{p}}
\]
and \( L^p([0, 1]) = \{ f : [0, 1] \to \mathbb{C} \mid f \) is measurable and \( \|f(x)\|_p < \infty \} \). Show that \( L^p([0, 1]) \) is a vector space.

3) Let \( m \) be Lebesgue measure on \( \mathbb{R} \) and suppose \( f \in L^1(m) \). Fix \( y \in \mathbb{R} \) and let \( g(x) = f(x - y) \). Show that \( g \) is integrable and that \( \int f \, dm = \int g \, dm \).

4) Let \( m \) be Lebesgue measure on \( \mathbb{R} \) and suppose \( f \in L^1(m) \). Fix \( c \in \mathbb{R} \), with \( c \neq 0 \), and let \( g(x) = f(cx) \). Show that \( g \) is integrable and find a formula for \( \int g \, dm \) in terms of \( \int f \, dm \).

5) Let \( m \) denote Lebesgue measure on \( \mathbb{R} \).

   i) Let \( E \subseteq \mathbb{R} \) and denote by \( E_2 \) the set \( \{(x, y) \mid x - y \in E\} \). Show that if \( E \) is a Borel set then \( E_2 \) is a Borel subset of \( \mathbb{R}^2 \).

ii) If \( f : \mathbb{R} \to \mathbb{C} \) is a Borel function and \( F(x, y) = f(x - y) \) then \( F : \mathbb{R}^2 \to \mathbb{C} \) is a Borel function.

iii) Suppose \( f, g \in L^1(\mathbb{R}) \). Let \( x \in \mathbb{R} \) and \( \phi_x(y) = f(y)g(x - y) \). Show that for almost all \( x \), \( \phi_x \) is integrable. For such \( x \) let \( \psi(x) = \int \phi_x \, dm \) and let \( \psi(x) = 0 \) if \( \phi_x \) is not integrable. Show
\[
\int |\psi| \, dm \leq \int |f| \, dm \int |g| \, dm.
\]