MATH 891 Analysis I Autumn 2017

Assignment 3, Due Nov. 1

1) Let $\{a_{m,n}\}_{m,n}$ be a double indexed sequence of non-negative real numbers. Let

$$S = \sup_F \sum_{(m,n)\in F} a_{m,n}$$

where the supremum runs over all *finite* subsets of $\mathbb{N}\times\mathbb{N}$. S may be finite or infinite.

i) Show that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{m,n} = S$$

Let $\{(m_k, n_k)\}_k$ be any enumeration of $\mathbb{N}\times\mathbb{N}$.

ii) Show that

$$S = \lim_{K\to\infty}\sum_{k=1}^{K} a_{m_k,n_k}$$

2) Guess the limits, as $n\to\infty$, of

$$\int_0^n \left(1-\frac{x}{n}\right)^n e^{x/2}\,dx \text{ and } \int_0^n \left(1+\frac{x}{n}\right)^n e^{-2x}\,dx$$

and prove that your guesses are correct.

3) Let X be a compct metric space and $C(X)$ the vector space of continuous complex valued functions on X. For $f,g\in C(X)$ and $\alpha\in\mathbb{C}$ we let

(*a*) $(f+g)(x) = f(x)+g(x)$;

(*b*) $(\alpha f)(x) = \alpha f(x)$;

(*c*) $(fg)(x) = f(x)g(x)$;

(*d*) $\|f\| = \sup_{x\in X}|f(x)|$;

(*e*) $\overline{f}(x) = \overline{f(x)}$.

Show that for $f,g\in C(X)$, $\alpha\in\mathbb{C}$

i) $\|f+g\|\le\|f\|+\|g\|$;

ii) $\|\alpha f\| = |\alpha|\|f\|$;

iii) $\|fg\|\le\|f\|\|g\|$;

iv) $\|\overline{f}f\| = \|f\|^2$.

4) Let us continue with the notations of exercise **3**. We say that $f\in C(X)$ is *positive* if $f(x)\ge 0$ for all $x\in X$ and that for $f,g\in C(X)$ real valued that $f\le g$ if $f(x)\le g(x)$ for all $x\in X$.

Let $\phi: C(X)\to\mathbb{C}$ be linear. If ϕ is such that $\phi(f)\ge 0$ whenever $f\ge 0$, we say that ϕ is *positive*. From **now** on suppose ϕ is positive.

Let $\langle f,g\rangle_\phi = \phi(f\overline{g})$ and $\|f\|_\phi^2 = \langle f,f\rangle_\phi$. Note that at this level of generality we might have $\|f\|_\phi = 0$ with $f\ne 0$. Show that

i) $\langle f,f\rangle_\phi\ge 0$;

ii) $\overline{\langle f,g\rangle_\phi} = \langle g,f\rangle_\phi$;

iii) $\langle f_1+f_2,g\rangle_\phi = \langle f_1,g\rangle_\phi + \langle f_2,g\rangle_\phi$;

iv) $\langle\alpha f,g\rangle_\phi = \alpha\langle f,g\rangle_\phi$;

v) if $\langle f,g\rangle_\phi = 0$ then $\|f+g\|_\phi^2 = \|f\|_\phi^2+\|g\|_\phi^2$;

vi) if $\|g\|_\phi = 0$ then $\langle f,g\rangle_\phi = 0$ for all f;

vii) $|\langle f, g\rangle_\phi| \leq \|f\|_\phi \|g\|_\phi$.

5) Let us continue with the notations of exercise **3**. Let 1 be the constant function equal to 1 for all $x \in X$. Show that

i) if f is positive then $f \leq \|f\| 1$;

ii) $|\phi(f)| \leq \|f\|\phi(1)$ for all $f \in C(X)$.