Assignment 4, Due Friday Dec. 2

1) If $1 < p < \infty$ show that the closed unit ball in $L^p(\mu)$ is strictly convex; i.e. if $f \neq g$ and $\|f\|_p = \|g\|_p = 1$, then $\|\frac{1}{2}(f + g)\|_p < 1$.

2) Let $H$ be a Hilbert space and $M \subset H$ a closed linear subspace. Let $x \in H$. Show that

$$\min\{|\langle x, m' \rangle| \mid m' \in M^\perp, \|m'\| = 1\} = \max\{\|x - m\| \mid m \in M\}.$$  

3) Let $\{e_n\}$ be an orthonormal set in a Hilbert space $H$.

i) Let $E = \{e_n\}$. Show that $E$ is closed and bounded but not compact.

ii) Let $F$ be the set of vectors in $H$ of the form $\sum_{n=1}^\infty c_n e_n$ with $c_n \in \mathbb{C}$ and $|c_n| \leq 1/n$ for all $n$. Show that $F$ is compact. (One way to do this is to use the concept of an $\epsilon$-net. Recall that a finite subset of $F$ is an $\epsilon$-net if for every $f \in F$ there is $a \in A$ such that $\|a - f\| < \epsilon$. A convenient characterization of compactness is that a set is compact if and only if for every $\epsilon > 0$ there is an $\epsilon$-net.)

iii) Show that $H$ is not locally compact.$^{(1)}$

4) Let $[0, 1]$ have Lebesgue measure $m$.

i) Let $f \in L^1[0, 1]$. Show that there exists $g \in L^\infty[0, 1]$ such that $g \neq 0$ and

$$\int fg \, dm = \|f\|_1\|g\|_\infty.$$

ii) Let $f \in L^\infty[0, 1]$ and $\epsilon > 0$ be given. Show that there is $g \in L^1[0, 1]$ such that $g \neq 0$ and

$$\left(\|f\|_\infty - \epsilon\right)\|g\|_1 \leq \int fg \, dm.$$

5) Let $E \subset [-\pi, \pi]$ be Lebesgue measurable. Show that

$$\lim_{n \to \infty} \int_E \sin(nx) \, dx = \lim_{n \to \infty} \int_E \cos(nx) \, dx = 0.$$  

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$^{(1)}$Rudin: Definition 2.3 (f) page 36.