MATH 891 Analysis I Autumn 2017

Assignment 4, Due Nov. 17

Let us recall the construction of the Riemann integral. Let $f : [a,b] \to \mathbb{R}$ be a bounded function. For a *partition* $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$ we set

$$\mathcal{L}(f,\mathcal{P}) = \sum_{i=0}^{n} m_i(x_i - x_{i-1}) \text{ and}$$

$$\mathcal{U}(f,\mathcal{P}) = \sum_{i=0}^{n} M_i(x_i - x_{i-1}) \text{ where}$$

$m_i = \inf_{x_{i-1}<x\leq x_i} f(x)$ and $M_i = \sup_{x_{i-1}<x\leq x_i} f(x)$. We then let

$$\overline{\int_a^b} f(x)\,dx = \inf_{\mathcal{P}} \mathcal{U}(f,\mathcal{P})$$

and

$$\underline{\int_a^b} f(x)\,dx = \sup_{\mathcal{P}} \mathcal{L}(f,\mathcal{P}).$$

Recall that f is Riemann integrable, *by definition*, if $\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx$.

1) Let $f : [a,b] \to \mathbb{R}$ be a bounded measurable function. Let $\bar{f}$ and $\underline{f}$ be as in Assignment 1 exercise 4, where they were denoted h and g respectively. Show that

i) $\overline{\int_a^b} f(x)\,dx = \int_{[a,b]} \bar{f}\,dm$;

(*Hint:* let $f_n(x) = \sup_{|x-y|<n^{-1}} f(y)$. Show that if $\mathcal{P}$ is a partition such that $x_i - x_{i-1} < n^{-1}$, then $\mathcal{U}(\mathcal{P}, f) \leq \int_{[a,b]} f_n\,dm$. Use this to show that $\overline{\int_a^b} f(x)\,dx \leq \int_{[a,b]} \bar{f}\,dm$.)

ii) $\underline{\int_a^b} f(x)\,dx = \int_{[a,b]} \underline{f}\,dm$;
[where the right hand integrals are interpreted as Lebesgue integrals]

iii) f is Riemann integrable if and only if f is continuous almost everywhere.

2) Suppose ϕ is continuous on $[a,b)$ and convex on (a,b).

i) Show that ϕ is bounded below on $[a,b)$.

ii) Show that $\lim_{t\to b^-} \phi(t)$ exits as an extended real number in $(-\infty,\infty]$.

3) Suppose that f is a non-negative measurable function on $[0,1]$ with Lebesgue measure. Suppose that $g(x) = \log(f(x))$ is integrable. Which of the two is smaller:

$$\int_{[0,1]} g\,dm \text{ or } \log\Big(\int_{[0,1]} f\,dm\Big)?$$

4) Suppose $(X,\mathfrak{M},\mu)$ is a measure space and $\{f_n\}_n$ is a sequence of real valued measurable functions. Suppose that g is an integrable function and for all n we have $|f_n(x)| \leq g(x)$ almost everywhere.
Show that

$$\int \underline{\lim}_n f_n\,d\mu \leq \underline{\lim}_n \int f_n\,d\mu \leq \overline{\lim}_n \int f_n\,d\mu \leq \int \overline{\lim}_n f_n\,d\mu$$

where $\underline{\lim}_n f_n = \liminf_n f_n$ and $\overline{\lim}_n f_n = \limsup_n f_n$.

5) Let $(X,\mathfrak{M},\mu)$ be a measure space and $p,q,r \geq 1$ be such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. For $f \in L^p(\mu)$ and $g \in L^q(\mu)$ show that

$$\|fg\|_r \leq \|f\|_p \|g\|_q.$$

November 15, 2017