Assignment 4, Due Nov. 17

Let us recall the construction of the Riemann integral. Let \( f : [a, b] \to \mathbb{R} \) be a bounded function. For a partition \( \mathcal{P} = \{ a = x_0 < x_1 < \cdots < x_n = b \} \) we set

\[
\mathcal{L}(f, \mathcal{P}) = \sum_{i=0}^{n} m_i(x_i - x_{i-1}) \quad \text{and} \quad \mathcal{U}(f, \mathcal{P}) = \sum_{i=0}^{n} M_i(x_i - x_{i-1})
\]

where

\[
m_i = \inf_{x_{i-1} < x \leq x_i} f(x) \quad \text{and} \quad M_i = \sup_{x_{i-1} < x \leq x_i} f(x).
\]

Then let

\[
\int_a^b f(x) \, dx = \inf_{\mathcal{P}} \mathcal{U}(f, \mathcal{P})
\]

and

\[
\int_a^b f(x) \, dx = \sup_{\mathcal{P}} \mathcal{L}(f, \mathcal{P}).
\]

Recall that \( f \) is Riemann integrable, by definition, if \( \int_a^b f(x) \, dx = \int_a^b f(x) \, dx \).

1) Let \( f : [a, b] \to \mathbb{R} \) be a bounded measurable function. Let \( f \) and \( g \) be as in Assignment 1 exercise 4, where they were denoted \( h \) and \( g \) respectively. Show that

\[
i) \quad \int_a^b f(x) \, dx = \int_{[a,b]} f \, dm;
\]

(Hint: let \( f_n(x) = \sup_{|x - y| < n^{-1}} f(y) \). Show that if \( \mathcal{P} \) is a partition such that \( x_i - x_{i-1} < n^{-1} \), then \( \mathcal{U}(\mathcal{P}, f) \leq \int_{[a,b]} f \, dm \). Use this to show that \( \int_a^b f(x) \, dx \leq \int_{[a,b]} f \, dm \).

\[
ii) \quad \int_a^b f(x) \, dx = \int_{[a,b]} f \, dm;
\]

(where the right hand integrals are interpreted as Lebesgue integrals)

\[
iii) \quad f \text{ is Riemann integrable if and only if } f \text{ is continuous almost everywhere.}
\]

2) Suppose \( \phi \) is continuous on \([a, b]\) and convex on \((a, b)\).

\[
i) \text{Show that } \phi \text{ is bounded below on } [a, b].
\]

\[
ii) \text{Show that } \lim_{t \to b^-} \phi(t) \text{ exits as an extended real number in } (-\infty, \infty).
\]

3) Suppose that \( f \) is a non-negative measurable function on \([0, 1]\) with Lebesgue measure. Suppose that \( g(x) = \log(f(x)) \) is integrable. Which of the two is smaller:

\[
\int_{[0,1]} g \, dm \text{ or } \log \left( \int_{[0,1]} f \, dm \right)?
\]

4) Suppose \((X, \mathcal{M}, \mu)\) is a measure space and \( \{f_n\}_n \) is a sequence of real valued measurable functions. Suppose that \( g(x) = \log(f(x)) \) is an integrable function and for all \( n \) we have \( |f_n(x)| \leq g(x) \) almost everywhere.

Show that

\[
\liminf_n f_n \, d\mu \leq \limsup_n f_n \, d\mu \leq \int f_n \, d\mu \leq \int \limsup_n f_n \, d\mu
\]

where \( \liminf_n f_n = \lim \inf_n f_n \) and \( \limsup_n f_n = \lim \sup_n f_n \).

5) Let \((X, \mathcal{M}, \mu)\) be a measure space and \( p, q, r \geq 1 \) be such that \( \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \). For \( f \in L^p(\mu) \) and \( g \in L^q(\mu) \) show that

\[
\|fg\|_r \leq \|f\|_p \|g\|_q.
\]