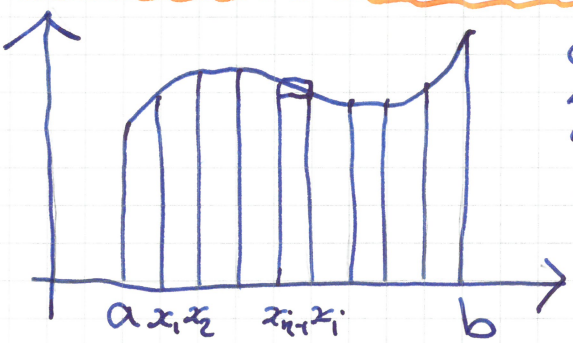
MATH 891
ANALYSIS I FALL 2017

MINGO • MATH & STATS
QUEEN'S UNIVERSITY

Riemann's Approach to Integration



$\{a < x_0 < x_1 < \cdots < x_n = b\} = P$

$m_i = \min_{x_{i-1} \le x \le x_i} f(x) \quad M_i = \max_{x_{i-1} \le x \le x_i} f(x)$

$\Delta x_i = x_i - x_{i-1}$

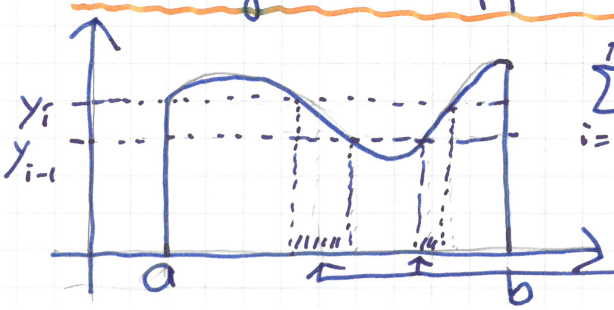
$$L_P = \sum_{i=1}^{n} m_i \Delta x_i \ , \quad U_P = \sum_{i=1}^{n} M_i \Delta x_i$$

If $\lim_{\mu(P) \to 0} L_P = \lim_{\mu(P) \to 0} U_P \quad \left(\mu(P) = \max_{1 \le i \le n} \Delta x_i\right)$

we set $\int_a^b f(x)\,dx$ to be this limit.

We call $\int_a^b f(x)\,dx$ the Riemann integral of f on $[a,b]$. Facts: this always exists for piecewise continuous functions. For these functions we can evaluate using the fundamental theorem of calculus.

Lebesgue's Approach to Integration



$\sum_{i=1}^{N} y_i \, m(f^{-1}(y_{i-1}, y_i))$ where

$m(E)$ is the "measure" of a set E

$f^{-1}(y_{i-1}, y_i)$.

② Here $y_0 < y_1 < \cdots < y_N$ is a partition of the y-axis. If $\lim_{\mu(P)\to 0} \sum_{i=1}^{N} y_i \bar{f}((y_{i-1}, y_i))$ converges we call the limit the Lebesgue integral of f.

Extended Real Numbers

$\overline{\mathbb{R}} = [-\infty, \infty] = \mathbb{R} \cup \{-\infty, \infty\}$.

- $\forall a \in \mathbb{R}$, $-\infty < a < \infty$, $-\infty < \infty$
- $a + \infty = \infty$, $a - \infty = -\infty$ $\forall a \in \mathbb{R}$
- $\infty + \infty = \infty$, $-\infty - \infty = -\infty$

Topology

X is a topological space if X is a set with a collection of subsets $\mathcal{T}$ satisfying some conditions.

- $\emptyset, X \in \mathcal{T}$
- $\{A_\alpha\}_\alpha \subseteq \mathcal{T} \Rightarrow \bigcup_\alpha A_\alpha \in \mathcal{T}$
- $A_1, \ldots, A_n \in \mathcal{T} \Rightarrow \bigcap_{i=1}^{n} A_i \in \mathcal{T}$

Elements of $\mathcal{T}$ are called the open sets of the topology. If $A \subseteq X$ and $X \setminus A \in \mathcal{T}$ then we say A is closed.

Relative Topology

If $(X, \mathcal{T})$ is a topological space and $Y \subseteq X$ is a subset, we may define a topology on Y by setting $\mathcal{T}_Y = \{A \cap Y \mid A \in \mathcal{T}\}$. We call $\mathcal{T}_Y$ the relative topology on Y. If $A \subseteq Y$ and $A \in \mathcal{T}_Y$ we say A is relatively open.

Examples

- $X = \mathbb{R}$, $\mathcal{T} = \{A \subseteq X \mid \forall a \in A \ \exists \ \varepsilon > 0$ such that $(a-\varepsilon, a+\varepsilon) \subseteq A\}$. $\mathcal{T}$ is the standard topology on $\mathbb{R}$.

- $Y = [0,1) \subseteq \mathbb{R}$ $A \subseteq Y$ is relatively open if $\forall a \in A \ \exists \ \varepsilon > 0$ such that (if $a > 0$ $(a-\varepsilon, a+\varepsilon) \subseteq A$ or if $a = 0$ $[0, \varepsilon) \subseteq A$).

- $X = \overline{\mathbb{R}}$. $A \subseteq X$ is open if $\forall a \in A \ \exists$

$$\begin{cases} \varepsilon > 0 \text{ s.t. } (a-\varepsilon, a+\varepsilon) \subseteq A & \text{if } a \in \mathbb{R} \\ M \in \mathbb{R} \quad (M, \infty] \subseteq A & a = \infty \\ M \in \mathbb{R} \quad [-\infty, M) \subseteq A & a = -\infty \end{cases}$$

④ # Metric Spaces

(X,d) is a metric space if X is a set and $d: X \times X \longrightarrow \mathbb{R}^+ = \{x \in \mathbb{R} \mid x \geq 0\}$, the metric, is such that

- $d(x,y) = d(y,x)$
- $d(x,y) = 0 \iff x = y$
- $d(x,y) + d(y,z) \leq d(x,z)$ (triangle inequality)

Every metric on X produces a topology on X, called the metric topology.

$\mathcal{T} = \{A \subseteq X \mid \forall a \in A \ \exists r > 0 \text{ s.t. } B(a,r) \subseteq A\}$

where $B(a,r) = \{b \in X \mid d(a,b) < r\}$.

$B(a,r)$ is called the open ball with centre a and radius r. One has to prove that an open ball is open!

Examples

Euclidean metric $\vec{x} = (x_1, \dots, x_n) \in \mathbb{R}^n, \mathbb{C}^n$

$d(\vec{x},\vec{y}) = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2}$.

Continuous Functions

Let X and Y be topological spaces, with

$\mathcal{T}_X$ denoting the open sets of X and $\mathcal{T}_Y$ denoting the open sets of Y. We say f is continuous if $\forall A \in \mathcal{T}_Y$ we have $f^{-1}(A) \in \mathcal{T}_X$: "The inverse image of an open set is open".

Continuity in Metric Spaces

If (X,d) and (Y,d) are metric spaces and $f: X \to Y$ the f is continuous in the metric topology if and only if whenever $\{x_n\}_n \subseteq X$ is a sequence converging to $x \in X$ we have $\{f(x_n)\}_n \subseteq Y$ converges to $f(x)$. We summarize this by saying f takes convergent sequences to convergent sequences.

Measure Spaces

Let X be a set and $\mathcal{M}$ a collection of subsets. We say that $\mathcal{M}$ is an algebra of sets if

- $\emptyset, X \in \mathcal{M}$
- $\mathcal{M}$ is closed under complements:
  $$E \in \mathcal{M} \Rightarrow E^c = X \setminus E \in \mathcal{M}$$

- $\mathcal{M}$ is closed under finite unions:
$$E_1, \dots, E_n \in \mathcal{M} \Rightarrow E_1 \cup \dots \cup E_n \in \mathcal{M}.$$

If, in addition, $\mathcal{M}$ is closed under countable unions, then we call $\mathcal{M}$ is a σ-algebra.

Remark

If $\mathcal{M}$ is a σ-algebra then

- $\mathcal{M}$ is closed under countable intersections: $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{M}$
$$\Rightarrow \bigcap_{n=1}^{\infty} A_n \in \mathcal{M}$$
- $\mathcal{M}$ is closed under relative complements:
$$A, B \in \mathcal{M} \Rightarrow A \setminus B = A \cap B^c \in \mathcal{M}$$

Example Let $n \in \mathbb{N}$ and $[n] = \{1, 2, 3, \dots, n\}$. For $1 \leq j \leq 2^n$ let $I_{n,j} = \left(\frac{j-1}{2^n}, \frac{j}{2^n}\right] \subseteq (0,1]$. Then $\bigcup_{j=1}^{2^n} I_{n,j} = (0,1]$ and $I_{n,j} \cap I_{n,k} = \emptyset$ for $j \neq k$. For $J \subseteq [n]$ let $E_J = \bigcup_{j \in J} I_{n,j}$. $(E_J)^c = E_{J^c}$, $E_{J_1} \cup E_{J_2} = E_{J_1 \cup J_2}$, $E_{J_1} \cap E_{J_2} = E_{J_1 \cap J_2}$. Conclusion: $\{E_J \mid J \subseteq [n]\}$ is

σ-algebra with 2^n subsets.

Given $0 < x \leq 1$ $\exists \{a_n\}_{n=1}^{\infty} \subseteq \{0,1\}$ such that $x = \sum_{n=1}^{\infty} \frac{a_n}{2^n}$. If we require that no sequence end in 0's then this expansion is unique. $1/2 = 0.1 = 0.0\dot{1}$

$5/8 = 0.100\dot{1}$ $I_{3,5} = \{0.100***\}$

j	1	2	3	4	5	6	7	8
exp	0.000	0.001	0.010	0.011	0.100	0.101	0.110	0.111

<u>Definition</u> (Measurable space)

$(X, \mathfrak{A})$ a set with a σ-algebra of subsets

MEASURABLE FUNCTIONS

Let $(X, \mathfrak{A})$ be a measurable space and $(Y, \mathcal{T})$ a topological space.

If $f: X \to Y$ is such that $f^{-1}(O) \in \mathfrak{A}$ for every $O \in \mathcal{T}$ we say f is <u>$\mathfrak{A}$-measurable</u> or just <u>measurable</u>.

Examples

- if $\mathfrak{A} = \{\emptyset, X\}$ then only the constant functions are $\mathfrak{A}$-measurable
- if $\mathfrak{A} = \mathcal{P}(X)$ = all subsets of X then every function is measurable
- if $X = (0,1]$ and $\mathfrak{A} = \mathfrak{A}_3$ then f is $\mathfrak{A}_3$ - measurable only if it only depends on a_1, a_2, a_3 as such a f must be constant on $I_{3,j}$, $j = 1, \dots, 2^3$.

Remark

If $(X, \mathfrak{A})$ and $(Y, \mathfrak{B})$ are measurable spaces and $f: X \to Y$ we don't define f to be measurable by saying $f^{-1}(B) \in \mathfrak{A}$ for all $B \in \mathfrak{B}$. However there is one instance in which this will be correct.