Riemann's Approach to Integration

\[ \int_{a}^{b} f(x) \, dx \leq \lim_{n \to \infty} L_p = \lim_{n \to \infty} \sum_{i=1}^{n} m_i \Delta x_i \leq \sum_{i=1}^{n} M_i \Delta x_i = U_p \geq \int_{a}^{b} f(x) \, dx \]

If \( \lim_{\mu(\mathcal{P}) \to 0} L_p = \lim_{\mu(\mathcal{P}) \to 0} U_p \) we set \( \int_{a}^{b} f(x) \, dx \) to be the limit. We call \( \int_{a}^{b} f(x) \, dx \) the Riemann integral of \( f \) on \([a, b]\). Facts: this always exists for piecewise continuous functions. For these functions we can evaluate using the fundamental theorem of calculus.

Lebesgue's Approach to Integration

\[ \int_{a}^{b} f(x) \, dx = \sum_{i=1}^{n} y_i \, m \left( \bigcap_{i=1}^{n} (y_{i-1}, y_i) \right) \]

where \( m(E) \) is the "measure" of set \( E \).
2. Here $y_0 < y_1 < \cdots < y_n$ is a partition of the $y$-axis. If $\lim_{m \to \infty} \sum_{i=1}^{n} y_i f(y_i, x_i)$ converges we call the limit the 
*Hilbert integral* of $f$.

**Extended Real Numbers**

$\mathbb{R}^* = [-\infty, \infty] = \mathbb{R} \cup \{-\infty, \infty\}$

- $\forall a \in \mathbb{R}^*$, $-\infty < a < \infty$, $-\infty < a < \infty$
- $a + \infty = \infty$, $a - \infty = -\infty$ for $a \in \mathbb{R}$
- $\infty + \infty = \infty$, $-\infty - \infty = -\infty$

**Topology**

$X$ is a topological space if $X$ is a set with a collection $\mathcal{T}$ of subsets $T$ satisfying some conditions.

- $\emptyset, X \in \mathcal{T}$
- $\forall A_k \in \mathcal{T} \Rightarrow \bigcup_{k} A_k \in \mathcal{T}$
- $A_1, \ldots, A_n \in \mathcal{T} \Rightarrow \bigcap_{i=1}^{n} A_i \in \mathcal{T}$

Elements of $\mathcal{T}$ are called the *open sets* of the topology. If $A \subseteq X$ and $X \setminus A \in \mathcal{T}$ then we say $A$ is *closed*. 
Relative Topology

If \((X, T)\) is a topological space and \(Y \subseteq X\) is a subset, we may define a topology on \(Y\) by setting \(T_Y = \{A \subseteq Y \mid A \in T\}\). We call \(T_Y\) the relative topology on \(Y\).

If \(A \subseteq Y\) and \(A \in T_Y\), we say \(A\) is relatively open.

Examples

1. \(X = \mathbb{R}, \quad T = \{A \subseteq X \mid \forall a \in A \exists \varepsilon > 0 \text{ such that } (a-\varepsilon, a+\varepsilon) \subseteq A\}\). \(T\) is the standard topology on \(\mathbb{R}\).

2. \(Y = [0,1] \subseteq \mathbb{R}, \quad A \subseteq Y\) is relatively open if \(\forall a \in A \exists \varepsilon > 0\) such that (if \(a > 0\) \((a-\varepsilon, a+\varepsilon) \subseteq A\) or if \(a = 0\) \([0,\varepsilon) \subseteq A\)).

3. \(X = \mathbb{R}, \quad A \subseteq X\) is open if \(\forall a \in A \exists \varepsilon > 0 \text{ s.t. } (a-\varepsilon, a+\varepsilon) \subseteq A\) if \(a \in \mathbb{R}\),
   \[
   \begin{cases} 
   \varepsilon > 0 \text{ s.t. } (a-\varepsilon, a+\varepsilon) \subseteq A & \text{if } a \in \mathbb{R} \\
   M \in \mathbb{R} \quad (M, \infty] \subseteq A & a = \infty \\
   M \in \mathbb{R} \quad [-\infty, M) \subseteq A & a = -\infty
   \end{cases}
   \]
Metric Spaces

$(X,d)$ is a metric space if $X$ is a set and $d : X \times X \rightarrow \mathbb{R}^+ = \{x \in \mathbb{R} | x \geq 0\}$, the metric, is such that

- $d(x,y) = d(y,x)$
- $d(x,y) = 0 \iff x = y$
- $d(x,y) + d(y,z) \leq d(x,z)$ (triangle inequality)

Every metric on $X$ produces a topology on $X$, called the metric topology.

$\mathcal{T} = \{A \subseteq X | \forall a \in A \exists r > 0 \text{ st. } B(a,r) \subseteq A\}$

where $B(a,r) = \{b \in X | d(a,b) < r\}$.

$B(a,r)$ is called the open ball with centre $a$ and radius $r$. One has to prove that an open ball is open!

Examples

Euclidean metric

$\tilde{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $d(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$.

Continuous Functions

Let $X$ and $Y$ be topological spaces, with
Fx denoting the open sets of X and Fy denoting the open sets of Y. We say f is continuous if \( \forall A \in F_y \) we have \( f^{-1}(A) \in F_x \). "The inverse image of an open set is open".

**Continuity in Metric Spaces**

If \((X,d)\) and \((Y,d)\) are metric spaces and \( f: X \rightarrow Y \) the \( f \) is continuous in the metric topology if and only if whenever \( \{x_n\}_{n \in \mathbb{N}} \subseteq X \) a sequence converging to \( x_0 \in X \) we have \( \{f(x_n)\}_{n \in \mathbb{N}} \subseteq Y \) converges to \( f(x_0) \). We summarize this by saying \( f \) takes convergent sequences to convergent sequences.

**Measure Spaces**

Let \( X \) be a set and \( \mathcal{M} \) a collection of subsets. We say that \( \mathcal{M} \) is an algebra of sets if

- \( \emptyset, X \in \mathcal{M} \)
- \( \mathcal{M} \) is closed under complements: \( E \in \mathcal{M} \Rightarrow E^c = X \setminus E \in \mathcal{M} \)
\( \mathcal{M} \) is closed under finite unions:

\[ E_1, ..., E_n \in \mathcal{M} \Rightarrow E_1 \cup \cdots \cup E_n \in \mathcal{M} \]

If, in addition, \( \mathcal{M} \) is closed under countable unions, then we call \( \mathcal{M} \) is a \( \cap \)-algebra.

**Remark**

If \( \mathcal{M} \) is a \( \cap \)-algebra then

- \( \mathcal{M} \) is closed under countable intersections:

\[ \bigcap_{n=1}^{\infty} A_n \in \mathcal{M} \]

- \( \mathcal{M} \) is closed under relative complements:

\[ A, B \in \mathcal{M} \Rightarrow A \setminus B = A \cap B^c \in \mathcal{M} \]

**Example**

Let \( n \in \mathbb{N} \) and \( [n] = \{1, 2, 3, \ldots, n\} \).

For \( 1 \leq j \leq 2^n \) let

\[ I_{n,j} = \left( \frac{j-1}{2^n}, \frac{j}{2^n} \right] \subseteq [0,1] \]

Then \( \bigcup_{j=1}^{2^n} I_{n,j} = [0,1] \) and \( I_{n,j} \cap I_{n,k} = \emptyset \) for \( j \neq k \). For \( J \subseteq [n] \) let

\[ E_J = \bigcup_{j \in J} I_{n,j} \]

\[ (E_J)^c = E_{J^c}, \quad E_J \cup E_{J^c} = E_{J \cup J^c}, \quad E_J \cap E_{J^c} = E_{J \cap J^c} \]

Conclusion: \( \{ E_J \mid J \subseteq [n] \} \) is
\[ \text{= algebra with } 2^n \text{ subsets.} \]

Given \( 0 < x \leq 1 \) \( \exists \sum_{n=1}^{\infty} a_n 2^n \in \{0,1\} \) such that \( x = \sum_{n=1}^{\infty} a_n 2^n \). If we require that no sequence ending in 01 then this expansion is unique. \( \frac{1}{2} = 0.1 = 0.01 \)

\( \frac{5}{8} = 0.1001 \) \( I_{35} = \{0.100**\} \)

\[ \begin{array}{cccccccc}
\hat{f} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
exp & 0.000 & 0.001 & 0.010 & 0.011 & 0.100 & 0.101 & 0.110 & 0.111 \\
\end{array} \]

**Definition (Measurable space)**

\((X, \mathcal{A})\) a set with a \( \sigma \)-algebra of subsets

**Measurable Functions**

Let \((X, \mathcal{A})\) be a measurable space and \((Y, \mathcal{T})\) a topological space.

If \( f : X \to Y \) is such that \( f^{-1}(O) \in \mathcal{A} \) for every \( O \in \mathcal{T} \) we say \( f \) is \( \mathcal{A} \)-measurable or just measurable.
Examples

- If $\mathcal{A}=3\emptyset, X^3$ then only the constant functions are $\mathcal{A}$-measurable.

- If $\mathcal{A}=\mathcal{P}(X)$ - all subsets of $X$ then every function is measurable.

- If $X=\{0,1\}$ and $\mathcal{A}=2^2$ then $f$ is $\mathcal{A}_2$-measurable only if it depends on $a_1, a_2, a_3$ as such $f$ must be constant on $I_{3,j}$, $j=0, \ldots, 2^3$.

Remark

If $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ are measurable spaces and $f: X \to Y$ we don't define $f$ to be measurable by saying $f^{-1}(B) \in \mathcal{A}$ for all $B \in \mathcal{B}$. However, there is one instance in which this will be correct.