For \( t < u < b \) we have for all \( a < s < b \)

\[
\frac{\phi(t) - \phi(s)}{t - s} \leq \frac{\phi(u) - \phi(t)}{u - t} \quad \text{So}
\]

\[
\beta \leq \frac{\phi(u) - \phi(t)}{u - t}.
\]

These two inequalities give us (**); so such a \( \beta \) exists and this concludes the proof.

**Examples** let \( \phi(x) = e^{\alpha x} \). For any probability space we have \( \exp(\int fdu) \leq \int e^{\alpha fdu} \)

for any integrable function \( f \). Suppose \( X = \{x_1, \ldots, x_n\} \), \( p_i = \mu(\{x_i\}) \). Then \( p_1 + \cdots + p_n = 1 \)

If we let \( y_i = e^{\alpha x_i} \), then \( \int fdu = \sum_{i=1}^{n} p_i x_i \) and

\[
\int e^{\alpha fdu} = \sum_{i=1}^{n} p_i x_i.
\]

So

\[
\chi_1 p_1 \cdots \chi_n p_n = \exp\left(\sum_{i=1}^{n} p_i x_i\right) = \exp(\int fdu) \leq \int e^{\alpha fdu} = p_1 y_1 + \cdots + p_n y_n.
\]

**Hölder & Minkowski's Inequalities**

Conjugate exponents: \( 1 < p, q < \infty \)

\[
\frac{1}{p} + \frac{1}{q} = 1.
\]

**WEEK 9**
Theorem: Let \((X, \mathcal{M}, \mu)\) be a measure space and \(f, g, X \to [0, \infty]\) measurable functions. Suppose \(p, q > 1\) and \(\frac{1}{p} + \frac{1}{q} = 1\). Then

\[
\int fg \, d\mu \leq \left( \int f^p \, d\mu \right)^{\frac{1}{p}} \left( \int g^q \, d\mu \right)^{\frac{1}{q}} \quad \text{(Hölder)}
\]

\[
\left( \int (f+g)^p \, d\mu \right)^{\frac{1}{p}} \leq \left( \int f^p \, d\mu \right)^{\frac{1}{p}} + \left( \int g^p \, d\mu \right)^{\frac{1}{p}} \quad \text{(Minkowski)}
\]

Proof: Let \(A = \left( \int f^p \, d\mu \right)^{\frac{1}{p}}\) \(B = \left( \int g^q \, d\mu \right)^{\frac{1}{q}}\).

If \(A = 0\) then \(f = 0\) a.e. so both equalities are trivial. So we may assume \(A, B > 0\). If either \(A = \infty\) or \(B = \infty\) then the R.H.S. = \(\infty\) so again the inequality is trivial.

Let us assume that \(0 < A, B < \infty\). Let \(F = \frac{f}{A}\) and \(G = \frac{g}{B}\). Then \(\int f^p \, d\mu = \int g^q \, d\mu = 1\). If we can show that \(\int FG \, d\mu \leq 1\) then \(\int fg \, d\mu = AB \int FG \, d\mu \leq AB\)

\[-\left( \int f^p \, d\mu \right)^{\frac{1}{p}} \left( \int g^q \, d\mu \right)^{\frac{1}{q}} \quad \text{as claimed.}
\]

Let \(x \in X\) and suppose \(0 < F(x) < \infty\) and \(0 < G(x) < \infty\). Write \(F(x) = e^{u/p}\) and
\[ G(x) = e^{\frac{x}{\alpha}}. \] Since \exp \text{ is convex and } \\
\frac{1}{p} + \frac{1}{q} = 1 \text{ we have } \exp(\frac{1}{p} s + \frac{1}{q} t) \leq \frac{1}{p} e^s + \frac{1}{q} e^t. \text{ Thus } F(x)G(x) = \exp(\frac{1}{p} s + \frac{1}{q} t) \leq \frac{1}{p} F(x)^p + \frac{1}{q} G(x)^q. \text{ If either } F(x) = 0 \text{ or } G(x) = 0, \text{ this inequality still holds. As } A < \infty, F(x) < \infty \text{ except possibly on a set of measure } 0; \text{ likewise for } G. \text{ Thus} \]

\[ F(x)G(x) \leq \frac{1}{p} F(x)^p + \frac{1}{q} G(x)^q \text{ a.e.} \]

Hence \[ \int F G \, du \leq \frac{1}{p} \int F^p \, du + \frac{1}{q} \int G^q \, du = \frac{1}{p} + \frac{1}{q} = 1. \]

Minkowskie’s Inequality

Let us write \((f+g)^p = f^p (f+g)^{p-1} + g^p (f+g)^{p-1}\) and apply Hölder’s inequality to each term.

\[ \int f (f+g)^{p-1} \, du \leq \left( \int f^p \, du \right)^{\frac{1}{p}} \left( \int (f+g)^{q(p-1)} \, du \right)^{\frac{1}{q-1}} = \left( \int f^p \, du \right)^{\frac{1}{p}} \left( \int (f+g)^{q(p-1)} \, du \right)^{\frac{1}{q-1}}. \text{ Likewise} \]

\[ \int g (f+g)^{p-1} \, du \leq \left( \int g^p \, du \right)^{\frac{1}{p}} \left( \int (f+g)^{q(p-1)} \, du \right)^{\frac{1}{q-1}} \]

Hence \[ \int (f+g)^p \, du \leq \left[ \left( \int f^p \, du \right)^{\frac{1}{p}} + \left( \int g^p \, du \right)^{\frac{1}{p}} \right] \left[ \left( \int (f+g)^{q(p-1)} \, du \right)^{\frac{1}{q-1}} \right] \]

or \[ \left( \int (f+g)^p \, du \right)^{\frac{1}{p}} = \left( \int (f+g)^p \, du \right)^{\frac{1}{p}} \leq \left( \int f^p \, du \right)^{\frac{1}{p}} + \left( \int g^p \, du \right)^{\frac{1}{p}} \]

provided there are no “irregularities”:

- \( \int (f+g)^p \, du = 0 \), or
- \( \int (f+g)^p \, du = \infty \).
If \( \int (f+g)^p \, du = 0 \) then \( (\int (f+g)^p \, du)^{1/p} = 0 \) and Minkowski's inequality holds trivially.

If \( \int (f+g)^p \, du = \infty \) we shall show that
\[
(\int f^p \, du)^{1/p} + (\int g^p \, du)^{1/p} = \infty,
\]
and again Minkowski's inequality holds, or equivalently
if \( (\int f^p \, du)^{1/p} + (\int g^p \, du)^{1/p} < \infty \) then
\[
\int (f+g)^p \, du < \infty.
\]
To see this note that \( \varphi(x) = x^p \) is a convex function for \( p > 1 \).

\[
\left( \frac{1}{2} f(x) + \frac{1}{2} g(x) \right)^p \leq \frac{1}{2} \left( f(x)^p + \frac{1}{2} g(x) \right)^p
\]
so
\[
(f(x)+g(x))^p \leq 2^{p-1} \left[ \int (f(x))^p + (g(x))^p \right]
\]
\[
\int (f(x)+g(x))^p \, du \leq 2^{p-1} \left[ \int f^p \, du + \int g^p \, du \right] < \infty
\]

**Complex Version**

For \( 1 < p < \infty \) and \( \frac{1}{p} + \frac{1}{q} = 1 \) we have
\[
\int |fg| \, du \leq \left( \int |f|^p \, du \right)^{1/p} \left( \int |g|^q \, du \right)^{1/q}.
\]

\[
\left( \int |f+g|^p \, du \right)^{1/p} \leq \left( \int (|f|+|g|)^p \, du \right)^{1/p} \leq \left( \int |f|^p \, du \right)^{1/p} \left( \int |g|^q \, du \right)^{1/q}
\]

**Equality**

If \( \alpha, \beta \in \mathbb{C} \) \( \alpha f = \beta g \) then we have equality in Minkowski's inequality and this is the only way.
$L^p$-spaces $1 \leq p \leq \infty$ (in particular $p = 1, 2, \infty$)

Let $(X, \mathcal{M}, \mu)$ be a measure space. If $f = g$ almost everywhere, we write $f \sim g$ and $[f]$ as the equivalence class of $f$. We let $L^{p, \infty}$

$L^p(\mu) = \{ [f] \mid f: X \rightarrow \mathbb{C}, \int |f|^p d\mu < \infty \}$

For $[f] \in L^p(\mu)$ we let $\|f\|_p = \left( \int |f|^p d\mu \right)^{1/p}$.

In abuse of notation we write $f$ instead of $[f]$. But we have remember that $f = 0$ means $f(x) = 0$ almost everywhere. If $X = \mathbb{R}$, $\mathcal{M} =$ Lebesgue measurable sets, $\mu =$ Lebesgue measure and $f$ is continuous then \( \{ x \mid \|f(x)\| > 0 \} \) is open and thus contains an open interval and thus has positive measure. Hence for $f$ and $g$ continuous functions $f = g$ a.e. implies that $f = g$ everywhere. Hence each equivalence class of $\sim$ contains at most one continuous function.

Notation: For $f \in L^p(\mu)$ we set $\|f\|_p = \left( \int |f|^p d\mu \right)^{1/p}$.
When we have Lebesgue measure on $\mathbb{R}$ we write $L^p(\mathbb{R})$. We can construct Lebesgue measure on $\mathbb{R}^n$ and we get $L^p(\mathbb{R}^n)$. If $X = \mathbb{N}$, we write $L^p(\mathbb{N})$. If $X = \mathbb{N}$ with counting measure we write

$$L^p(\mathbb{N}) = \{ (a_i)_{i \in \mathbb{N}} \mid \sum |a_i|^p < \infty \}.$$ 

If $X = \mathbb{Z}$ with counting measure we set

$$L^p(\mathbb{Z}) = \{ (a_i)_{i = -\infty}^{\infty} \mid \sum_{i = -\infty}^{\infty} |a_i|^p < \infty \}.$$ 

We have

$$\| (a_i)_{i = -\infty}^{\infty} \|_p = \left( \sum_{i = -\infty}^{\infty} |a_i|^p \right)^{\frac{1}{p}}.$$ 

For all of these spaces we have

$$\| fg \|_1 \leq \| f \|_p \| g \|_q, \quad \frac{1}{p} + \frac{1}{q} = 1$$

$$\| f + g \|_p \leq \| f \|_p + \| g \|_p, \quad 1 < p < \infty$$

**Essentially Bounded Functions**

Let $(X, \mathcal{M}, \mu)$ be a measure space and $f : X \to \mathbb{C}$ be measurable on $X$. If $f$ is essentially bounded, then $\mu\{ |f| > M \}$ for some $M > 0$. The $L^p$ norm of $f$ is given by

$$\| f \|_p = \left( \int |f|^p \, d\mu \right)^{\frac{1}{p}}.$$
$f: X \to [0, \infty]$. If $\exists M > 0$ such that $\{x \mid |f(x)| > M\}$ is a null set we say that $M$ is an essential bound of $f$, and that $f$ is essentially bounded. If $f$ is essentially bounded we set $\|f\|_\infty = \inf \{M \mid M$ is an essential bound of $f\}$. Note that $\inf \{a \mid a \in \mathbb{R}\} = -\infty$ and $\sup \{a \mid a \in \mathbb{R}\} = \infty$. So if $f$ is not essentially bounded $\|f\|_\infty = \infty$. Also $\{x \mid |f(x)| > \|f\|_\infty\} = \bigcup_{n=1}^{\infty} \{x \mid |f(x)| > \frac{1}{n} + \|f\|_\infty\}$ and each set on the right is a null set. Thus $\{x \mid |f(x)| > \|f\|_\infty\}$ is a null set and $\|f\|_\infty$ is itself an essential bound.

We declare 1 & $\infty$ to be conjugate exponents.

**Theorem** If $p$ & $q$ are conjugate exponents $f \in L^p$, $g \in L^q$ then $\|fg\|_1 \leq \|f\|_p \|g\|_q$. If $f_1, f_2 \in L^p$ then $\|f_1 + f_2\|_p \leq \|f_1\|_p + \|f_2\|_p$. 
Proof: When \( p = 1, q = \infty \) we have
\[
|f(x)| \leq \|f\|_\infty \quad \text{a.e.}
\]
so
\[
\|fg\|_1 = \int |fg| \, du \leq \int |f| \|g\|_\infty \, du = \|f\|_1 \|g\|_\infty
\]
\[
\|f_1 + f_2\|_1 = \int |f_1 + f_2| \, du \leq \int |f_1| + |f_2| \, du = \|f_1\|_1 + \|f_2\|_1
\]
If \( f_1, f_2 \in L^\infty \) then \( |f_1(x)| \leq \|f_1\|_\infty \) a.e.
and \( |f_2(x)| \leq \|f_2\|_\infty \) a.e. so
\[
|f_1(x) + f_2(x)| \leq |f_1(x)| + |f_2(x)| \leq \|f_1\|_\infty + \|f_2\|_\infty
\]
a.e. Hence \( \|f_1\|_\infty + \|f_2\|_\infty \) is an essential bound for \( f_1 + f_2 \). Thus
\[
\|f_1 + f_2\|_\infty \leq \|f_1\|_\infty + \|f_2\|_\infty
\]

Theorem: Suppose \( 1 \leq p \leq \infty \) and \( f, g \in L^p \),
but \( d(f, g) = \|f - g\|_p \). Then \((L^p, d)\) is a complete metric space.

Proof: By Minkowski's inequality \( d \) satisfies the triangle inequality. The other two properties for being a metric are easy. So the main task is to prove completeness. We begin by supposing \( 1 \leq p < \infty \).
Let \( \{f_n\}_{n \in \mathbb{N}} \subseteq L^p \) be a Cauchy sequence.

We may suppose, by passing to a subsequence if necessary, that \( \|f_{n+1} - f_n\|_p < \frac{2^n}{2} \).

Let \( g_k = \sum_{n=1}^{k} |f_{n+1} - f_n| \) and \( g = \sum_{n=1}^{\infty} |f_{n+1} - f_n| \).

We have \( \|g_k\|_p \leq \sum_{n=1}^{k} \|f_{n+1} - f_n\|_p \leq \sum_{n=1}^{\infty} \frac{2^n}{2} < 1 \).

Hence for all \( k \), \( \int g_k \, dx < 1 \). By construction \( \int g_k(x)^p \, dx \) in creases to \( g(x) \), thus \( \int (g_k(x))^p \, dx \) in creases to \( (g(x))^p \).

By Fatou's lemma we have

\[
\liminf_{k \to \infty} \int g_k^p \, dx \leq \int \liminf_{k \to \infty} g_k^p \, dx \leq 1.
\]

Hence \( \int g^p \, dx \leq 1 \) and so \( \mathcal{g}(x) < \infty \) almost everywhere. Thus \( \sum_{n=1}^{\infty} |f_{n+1}(x) - f_n(x)| \) converges almost everywhere; thus

\[
\sum_{n=1}^{\infty} f_{n+1}(x) - f_n(x) \quad \text{converges almost everywhere.}
\]

Now \( f_{n+1}(x) = f(x) + \sum_{k=1}^{n} f_{k+1}(x) - f_k(x) \). Thus

\[ (*) \quad \text{This is because of the following general fact: if a subsequence of a Cauchy sequence converges to } f; \text{ then the whole sequence converges to } f. \]
\[ \{ f_n(x) \}_{n=1}^{\infty} \text{ converges almost everywhere.} \]

Let \( f(x) = \lim_{n \to \infty} f_n(x) \).

**Claim:** \( \lim_{n \to \infty} \| f_n - f \|_p = 0 \). Let \( \epsilon > 0 \) be given; choose \( N \) s.t. \( \| f_m - f_{n} \|_p < \epsilon \) for \( m, n > N \). Then

\[
\int |f - f_n|^p \, du = \int \lim_{n \to \infty} |f_n - f_n|^p \, du = \liminf_{n \to \infty} \int |f_k - f_n|^p \, du
\]

\[
\leq \liminf_{k \to \infty} \int |f_k - f_n|^p \, du < \epsilon^p \text{ for } n > N.
\]

Thus \( \lim_{n \to \infty} \| f - f_n \|_p = 0 \). In particular, \( f - f_n \in L^p \) so \( f = (f - f_n) + f_n \in L^p \).

Hence \( L^p \) is complete.

**The Case of \( L^\infty \)**

Suppose \( \{ f_n \}_{n=1}^{\infty} \) is a Cauchy sequence in \( L^\infty(\mu) \). For each \( n \) let \( A_n = \{ x \mid |f_n(x)| > \| f_n \|_{\infty} \} \), and \( B_{m,n} = \{ x \mid |f_m(x) - f_n(x)| > \| f_m - f_n \|_{\infty} \} \).

Then \( \mu(A_n) = 0 \) and \( \mu(B_{m,n}) = 0 \) for all \( m, n \).

Let \( E = \bigcup_{n=1}^{\infty} A_n \cup \bigcup_{m,n=1}^{\infty} B_{m,n} \). Then \( \mu(E) = 0 \).

For \( x \notin E \) \( |f_n(x) - f_m(x)| \leq \| f_m - f_n \|_{\infty} \) so \( \{ f_n(x) \}_{n=1}^{\infty} \) is a Cauchy sequence in \( C \).

For \( x \in E \) let \( f(x) = 0 \) and for \( x \notin E \) let \( f(x) = \lim_{n \to \infty} f_n(x) \).
Note that $\|f_m\|_{\infty} - \|f\|_{\infty} \leq \|f_m - f\|_{\infty}$
so $\{\|f_m\|_{\infty} \}_{m=1}^\infty \subseteq E$ is also a Cauchy
sequence and thus converges and thus
is bounded. So let $M > 0$ be such that
$\|f\|_{\infty} \leq M \forall n$. Then for all $x$, $|f(x)|$
$\leq M$ because either $x \in E$ and $f(x) = 0$,
or $x \notin E$ and $|f(x)| = \lim_{n} |f_n(x)| \leq \lim_{n} \|f_n\|_{\infty}$
$\leq M$. So $f \in L^\infty$,

Let $\varepsilon > 0$ be given and let us show
that there is $N$ such that for $n > N$, $\|f_n - f\|_{\infty} < \varepsilon$.
Choose $N$ such that $\|f_m - f\|_{\infty} < \varepsilon$ for
$m < N$. Suppose $x \notin E$ and $n > N$ then
$|f(x) - f_n(x)| = \lim_{m} |f_m(x) - f_n(x)|$
$\leq \limsup_{m} \|f_m - f\|_{\infty} \leq \varepsilon$. Hence
$\|f - f_n\|_{\infty} < \varepsilon$ for $n > N$.

Remark: We have just seen that
given a convergent sequence $\{f_n\}_{n=1}^\infty \subseteq L^p(E)$
there is a subsequence which converges
almost surely. When $p = \infty$ we don't have
to pass to a subsequence.
Theorem: Let $1 \leq p < \infty$ and $(X, M, \mu)$ be a measure space. Let $S = \{f: x \rightarrow C \mid f$ is measurable, $f(x)$ is a finite set, and $\mu(\{x \mid |f(x)| > 0\}) < \infty\}$. Then $S$ is a dense subspace in $L^p(\mu)$.

Proof: Let $f \in S$ and $E = \{x \mid |f(x)| > 0\}$. Let $M$ be the largest, in modulus, of $f(x)$. Then for $x \in E$, $|f(x)| \leq M$ and for $x \notin E$, $f(x) = 0$. Thus $\int |f|^p \, d\mu = \int_E |f|^p \, d\mu \leq M^p \mu(E) < \infty$. Hence $f \in L^p(\mu)$. Thus $S \subseteq L^p(\mu)$. It is routine to check that $S$ is a subspace. Let $f \in L^p(\mu)$ be such that $f \geq 0$. Let $\{s_n\}_{n=1}^{\infty}$ be a sequenced simple functions increasing pointwise to $f$. Then $0 \leq s_n \leq f$ and $s_n \in S$ because $0 \leq s_n^p \leq f^p$ and $\int s_n^p \, d\mu \leq \int f^p \, d\mu < \infty$. Also $\int |f - s_n|^p \, d\mu \leq \int f^p \, d\mu < \infty$. So by the dominated convergence theorem
\[ \lim_{n \to \infty} \int f - s_n^P \, du = 0; \text{ hence } \| f - s_n \|_P \to 0. \]

For general \( f \), write \( f = u + iv \) with \( u, v \) real and then decompose \( u \) & \( v \) into positive and negative parts. Find a sequence \( s_n \) for each part and then take \( s_n = s_{n, u} - s_{n, v} + i(s_{n, v} - i s_{n, u}) \).

**Theorem**

For \( 1 \leq p < \infty \), \( C_c(X) \) is dense in \( L^p(\mu) \). \( X \) is a locally compact Hausdorff space.

**Proof:** (When \( X = \mathbb{R} \))

Let \( S \) be as in the previous theorem. Let \( s \in S \) and \( \varepsilon > 0 \) be given. By Hausdorff's theorem, there is \( g \in C_c(X) \) such that \( g = s \) except possibly on a set \( E \) of measure less than \( \varepsilon \), and \( \| g \|_\infty \leq \| s \|_\infty \). Thus for \( x \notin E \)

\[ \| g(x) - s(x) \|^p = 0 \] and for \( x \in E \)

\[ \| g(x) - s(x) \|^p \leq (2 \| s \|_\infty)^p. \]

Hence

\[ \int \| g - s \|^p \, d\mu = \int_E \| g - s \|^p \\mu \leq \mu(E) (2 \| s \|_\infty)^p. \]

So

\[ \| g - s \|_p \leq \varepsilon \cdot 2^{-\frac{p}{p'}} \| s \|_\infty. \]