For $t < u < b$ we have for all $a < s < b$

$$\frac{\varphi(t) - \varphi(s)}{t-s} \leq \frac{\varphi(u) - \varphi(t)}{u-t}$$

so

$$\beta \leq \frac{\varphi(u) - \varphi(t)}{u-t}.$$ 

These two inequalities give us (**); so such a $\beta$ exists and this concludes the proof.

Examples: let $\varphi(x) = e^x$. For any probability space we have $\exp(\int f \, du) = \int e^f \, du$ for any integrable function $f$. Suppose $X = \{x_1, \ldots, x_n\}$, $p_i = u(\{x_i\})$. Then $p_1 + \ldots + p_n = 1$.

If we let $x_i = e^{x_i}$, then $\int f \, du = \sum_{i=1}^{n} p_i x_i$ and $\int e^f \, du = \sum_{i=1}^{n} p_i \, e^{x_i}$. So

$$x_1^{p_1} \cdots x_n^{p_n} = \exp\left(\sum_{i=1}^{n} p_i \, x_i\right) = \exp(\int f \, du) \leq \int e^f \, du$$

$$= p_1 x_1 + \cdots + p_n x_n.$$ 

Hölder & Minkowski's Inequalities

Conjugate exponents: $1 < p, q < \infty$

$$\frac{1}{p} + \frac{1}{q} = 1.$$
Theorem: Let \((X, \mathcal{M}, \mu)\) be a measure space and \(f, g : X \to [0, \infty]\) measurable functions. Suppose \(p, q > 1\) and \(\frac{1}{p} + \frac{1}{q} = 1\). Then
\[
\int f g \, d\mu \leq (\int f^p \, d\mu)^{\frac{1}{p}} (\int g^q \, d\mu)^{\frac{1}{q}} \quad \text{(Hölder)}
\]
\[
(\int (f+g)^p \, d\mu)^{\frac{1}{p}} \leq (\int f^p \, d\mu)^{\frac{1}{p}} + (\int g^p \, d\mu)^{\frac{1}{p}} \quad \text{(Minkowski)}
\]

Proof: Let \(A = (\int f^p \, d\mu)^{\frac{1}{p}}\), \(B = (\int g^q \, d\mu)^{\frac{1}{q}}\).

If \(A = 0\) then \(f = 0\) a.e. so both equalities are trivial. So we may assume \(A, B > 0\). If either \(A = \infty\) or \(B = \infty\) then the RHS = \(\infty\) so again the inequality is trivial. So let us assume that \(0 < A, B < \infty\). Let \(F = f/A\) and \(G = g/B\). Then
\[
\int F^p \, d\mu = \int G^q \, d\mu = 1. \text{ If we can show that } \int FG \, d\mu \leq 1 \text{ then } \int fg \, d\mu = AB \int f g \, d\mu \leq AB^2 (\int f^p \, d\mu)^{\frac{1}{p}} (\int g^q \, d\mu)^{\frac{1}{q}} \text{ as claimed.}
\]

Let \(x \in X\) and suppose \(0 < F(x) < \infty\) and \(0 < G(x) < \infty\). Write \(F(x) = e^{\frac{p}{p}} \text{ and } \)
$\text{Let } \Gamma(x) = e^{t/s}$. Since $\exp$ is convex and
\[ \frac{1}{p} + \frac{1}{q} = 1 \]
we have $\exp(\frac{1}{p}s + \frac{1}{q}t) \leq \frac{1}{p} e^s + \frac{1}{q} e^t$. Thus $F(x)G(x) = \exp(\frac{1}{p}s + \frac{1}{q}t)$
\[ \leq \frac{1}{p} F(x)^p + \frac{1}{q} G(x)^q \quad \text{if} \quad \text{either } F(x) = 0 \text{ or } G(x) = 0, \text{ then inequality still holds. As } A < \infty, F(x) < \infty \text{ except possibly on a set of measure } 0; \text{ likewise for } G. \text{ Thus}
\[ F(x)G(x) \leq \frac{1}{p} F(x)^p + \frac{1}{q} G(x)^q \quad \text{a.e.}
\]
Hence $\int F(x)G(x) dx \leq \frac{1}{p} \int F(x)^p dx + \frac{1}{q} \int G(x)^q dx = \frac{1}{p} + \frac{1}{q} = 1$.

\textbf{Minkowski's Inequality}

Let us write $(f + g)^p = f(fg)^{p-1} + g(fg)^{p-1}$ and apply Hölder's inequality to each term
\[ \int (f+g)^p dx \leq \left( \int f^p dx \right)^{\frac{1}{p}} \left( \int (fg)^{(p-1)} dx \right)^{\frac{1}{p-1}} \]
\[ = \left( \int f^p dx \right)^{\frac{1}{p}} \left( \int (f+g)^q dx \right)^{\frac{1}{q}}. \]

Likewise
\[ \int (f+g)^q dx \leq \left( \int g^q dx \right)^{\frac{1}{q}} \left( \int (f+g)^p dx \right)^{\frac{1}{p}} \]
Hence $\int (f+g)^p dx \leq \left\{ \left( \int f^p dx \right)^{\frac{1}{p}} + \left( \int g^q dx \right)^{\frac{1}{q}} \right\} \left( \int (f+g)^q dx \right)^{\frac{1}{q}}$
\[ \text{or } \left( \int (f+g)^p dx \right)^{\frac{1}{p}} = \left( \int (f+g)^q dx \right)^{\frac{1}{q}} \leq \left( \int f^p dx \right)^{\frac{1}{p}} + \left( \int g^q dx \right)^{\frac{1}{q}} \]
provided there are no "irregularities":
\cdot $\int (f+g)^p dx = 0$, or
\cdot $\int (f+g)^q dx = \infty$. 
If $\int (f+g)^p \, du = 0$ then $(\int (f+g)^p \, du)_{\infty} = 0$ and Minkowski's inequality holds trivially.

If $\int (f+g)^p \, du = \infty$ we shall show that

$(\int f^p \, du)^{\frac{1}{p}} + (\int g^p \, du)^{\frac{1}{p}} = \infty$, and again Minkowski's inequality holds, or equivalently

if $(\int f^p \, du)^{\frac{1}{p}} + (\int g^p \, du)^{\frac{1}{p}} < \infty$ then $\int (f+g)^p \, du < \infty$. To see this note that $\varphi(x) = x^p$ is a convex function for $p > 1$.

$\left(\frac{1}{2} f(x) + \frac{1}{2} g(x)\right)^p \leq \frac{1}{2} \left( f(x)^p + \frac{1}{2} g(x) \right)^p$ so

$\left( f(x)+g(x) \right)^p \leq 2^{p-1} \left( \int f^p \, du + \int g^p \, du \right)$

$\int (f(x)+g(x))^p \, du \leq 2^{p-1} \left( \int f^p \, du + \int g^p \, du \right) < \infty$

**Complex Version**

For $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ we have

$\int \left| f g \right| \, du \leq \left( \int |f|^p \, du \right)^{\frac{1}{p}} \left( \int |g|^q \, du \right)^{\frac{1}{q}}$.

$(\int |f+g|^p \, du)^{\frac{1}{p}} \leq (\int (|f|+|g|)^p \, du)^{\frac{1}{p}} \leq (\int |f|^p \, du)^{\frac{1}{p}} \left( \int |g|^q \, du \right)^{\frac{1}{q}}$

**Equality**

If $\alpha, \beta \in \mathbb{C}$ & $\alpha f = \beta g$ then we have equality in Minkowski's inequality and this is the only way.
Let \((X, \mathcal{M}, \mu)\) be a measure space. If \(f = g\) almost everywhere we write \(f \sim g\) and \([f]\) as the equivalence class of \(f\). We let \(1 \leq p < \infty\)
\[L^p(\mu) = \{ [f] \mid f: X \to \mathbb{C}, \int_X |f|^p \, d\mu < \infty \} \]
For \([f] \in L^p(\mu)\) we let \([f]_{L^p} = (\int_X |f|^p \, d\mu)^{\frac{1}{p}}\).
In abuse of notation we write \(f\) instead of \([f]\). But we have remembered that \(f = 0\) means \(f(x) = 0\) almost everywhere. If \(X = \mathbb{R}^n\), \(\mathcal{M}\) = Lebesgue measurable sets, \(\mu\) = Lebesgue measure and \(f\) is continuous then \(\{ x \mid |f(x)| > 0 \}\) is open and thus contains an open interval and thus has positive measure. Hence for \(f\) and \(g\) continuous functions \(f = g\) a.e. implies that \(f = g\) everywhere. Hence each equivalence class of \(\sim\) contains at most one continuous function.

Notation: For \(f \in L^p(\mu)\) we set \(||f||_p = (\int_X |f|^p \, d\mu)^{\frac{1}{p}}\).
When we have Lebesgue measure on \( \mathbb{R} \) we write \( L^p(\mathbb{R}) \). We can construct Lebesgue measure on \( \mathbb{R}^n \) and we get \( L^p(\mathbb{R}^n) \). If \( X = \mathbb{N} \) and \( \mu \) a counting measure we write
\[
L^p(\mathbb{N}) = \{ (a_i)_{i \in \mathbb{N}} \mid \sum_{i=1}^{\infty} |a_i|^p < \infty \},
\]
If \( X = \mathbb{Z} \) with counting measure we set
\[
L^p(\mathbb{Z}) = \{ (a_i)_{i=-\infty}^{\infty} \mid \sum_{i=-\infty}^{\infty} |a_i|^p < \infty \}
\]
we have
\[
\| (a_i)_{i=-\infty}^{\infty} \|_p = \sqrt[p]{\sum_{i=-\infty}^{\infty} |a_i|^p}
\]
and
\[
\| (a_i)_{i=-\infty}^{\infty} \|_p = \sqrt[p]{\sum_{i=-\infty}^{\infty} |a_i|^p}.
\]
For all of these spaces we have
\[
\| fg \|_1 \leq \| f \|_p \| g \|_q, \quad \frac{1}{p} + \frac{1}{q} = 1
\]
\[
\| f + g \|_p \leq \| f \|_p + \| g \|_p \quad 1 < p < \infty
\]

**Essentially Bounded Functions**

Let \((X, \mathcal{M}, \mu)\) be a measure space and \( f: X \rightarrow \mathbb{C} \) be measurable on
$f: X \to [0, \infty]$. If $M > 0$ such that $\{x | |f(x)| > M\}$ is a null set we say that $M$ is an **essential bound** of $f$, and that $f$ is **essentially bounded**. If $f$ is essentially bounded we set $\|f\|_\infty = \inf \{M | M$ is an essential bound of $f\}$. Note that 
\[ \inf \{a | a \notin \emptyset\} = \infty \quad \text{and} \quad \sup \{a | a \notin \emptyset\} = -\infty. \]
So if $f$ is not essentially bounded $\|f\|_\infty = \infty$. Also 
\[ \{x | |f(x)| > \|f\|_\infty\} = \bigcup_{n=1}^{\infty} \{x | |f(x)| > \frac{1}{n} + \|f\|_\infty\} \]
and each set on the right is a null set thus $\{x | |f(x)| > \|f\|_\infty\}$ is a null set and $\|f\|_\infty$ is itself an essential bound.

We declare $1$ & $\infty$ to be conjugate exponents.

**Theorem** If $p & q$ are conjugate exponents \(1 < p, q < \infty\) then \(\|fg\|_1 \leq \|f\|_p \|g\|_q\). If \( f, f_1, f_2 \in L^p \) then \(\|f_1 + f_2\|_p \leq \|f_1\|_p + \|f_2\|_p\).
Proof: When $p=1, q=\infty$ we have
\[ \|fg\|_{1} \leq \|fg\|_{\infty} \text{ a.e.} \]
\[ \|fg\|_{1} = \int |fg| \, dx \leq \int |f| \cdot \|g\|_{\infty} \, dx = \|f\|_{1} \cdot \|g\|_{\infty} \]
\[ \|f_{1}+f_{2}\|_{1} = \int |f_{1}+f_{2}| \, dx \leq \int |f_{1}|+|f_{2}| \, dx = \|f_{1}\|_{1}+\|f_{2}\|_{1} \]
If $f_{1}, f_{2} \in L^{\infty}$ then $|f_{1}(x)| \leq \|f_{1}\|_{\infty} \text{ a.e.}$ and $|f_{2}(x)| \leq \|f_{2}\|_{\infty} \text{ a.e.}$ so
\[ |f_{1}(x)+f_{2}(x)| \leq |f_{1}(x)|+|f_{2}(x)| \leq \|f_{1}\|_{\infty}+\|f_{2}\|_{\infty} \text{ a.e.} \]
Hence $\|f_{1}\|_{\infty}+\|f_{2}\|_{\infty}$ is an essential bound for $f_{1}+f_{2}$. Thus
\[ \|f_{1}+f_{2}\|_{\infty} \leq \|f_{1}\|_{\infty}+\|f_{2}\|_{\infty} \]

Theorem: Suppose $1 \leq p \leq \infty$ and $f, g \in L^{p}$. Let $d(f,g) = \|f-g\|_{p}$. Then $(L^{p}, d)$ is a complete metric space.

Proof: By Minkowski's inequality $d$ satisfies the triangle inequality. The other two properties for being a metric are easy. So the main task is to prove completeness. We begin by supposing $1 \leq p < \infty$. 
let \( \{f_n\}_{n \geq 1} \subseteq L^p \) be a Cauchy sequence. We may suppose, by passing to a subsequence if necessary, that \( \|f_{n+1} - f_n\|_p < 2^n \).

Let \( g_k = \sum_{n=k}^{\infty} |f_{n+1} - f_n| \) and \( g = \sum_{n=1}^{\infty} |f_{n+1} - f_n| \).

We have \( \|g_k\|_p \leq \sum_{n=k}^{\infty} \|f_{n+1} - f_n\|_p \leq \sum_{n=k}^{\infty} 2^n < 1 \). Hence for all \( k \), \( \int g_k \, du < 1 \). By construction \( \int g_k(x) \, dx \) in creases to \( g(x) \), thus \( \{g_k(x)\}_{k=1}^{\infty} \) in creases to \( (g(x))^\alpha \).

By Fatou's lemma we have

\[
\liminf_{k \to \infty} \int g_k \, du \leq \liminf_{k \to \infty} \int g_k^\alpha \, du \leq 1.
\]

Hence \( \int g^\alpha \, du \leq 1 \) and so \( 0 \leq g(x) < \infty \) almost everywhere. Thus \( \sum_{n=1}^{\infty} |f_{n+1}(x) - f_n(x)| \) converges almost everywhere; thus

\[
\sum_{n=1}^{\infty} f_{n+1}(x) - f_n(x)
\]

converges almost everywhere. Now \( f_{n+1}(x) = f(x) + \sum_{k=1}^{n} f_{k+1}(x) - f_k(x) \). Thus

(*) This is because of the following general fact: if a subsequence of a Cauchy sequence converges to \( f \), then the whole sequence converges to \( f \).
\( \{ f_n(x) \}_{n=1}^{\infty} \) converges almost everywhere.

Let \( f(x) = \lim_{n \to \infty} f_n(x) \).

**Claim:** \( \lim_{n \to \infty} \| f_n - f \|_p = 0 \). Let \( \varepsilon > 0 \) be given; choose \( N \) s.t. \( \| f_m - f_n \|_p < \varepsilon \) for \( m,n > N \). Then

\[
\int |f - f_n|^p \, du = \int \lim_{n \to \infty} |f_n - f|^p \, du = \lim_{n \to \infty} \int |f_k - f_n|^p \, du
\]

\( \leq \liminf \int |f_k - f|^p \, du < \varepsilon^p \) for \( n > N \).

Thus \( \lim_{n \to \infty} \| f - f_n \|_p = 0 \). In particular, \( f - f_n \in L^p \) so \( f = (f - f_n) + f_n \in L^p \).

Hence \( L^p \) is complete.

**The Case of** \( L^\infty \)