Math 892: Core Course in Analysis II, Winter 2010

Assignment 4, Due April 9

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1) Let H be a Hilbert space and $T : H \to H$ be a *linear* mapping, but not necessarily bounded. Show that T is bounded if

$$\sup\{|\langle Tx, x\rangle| \mid \|x\| \le 1\} < \infty.$$

2) Let $T \in \mathcal{L}(H)$ be self-adjoint.

a) Show that either $\|T\|$ or $-\|T\|$ is in the spectrum of T.

b) Let $r(T) = \sup\{|\lambda| \mid \lambda \in \sigma(T)\}$. Show that for T self-adjoint, $\|T\| = r(T)$

3) Let S and T be self-adjoint operators on H.

a) Suppose $S, T \ge 0$, show that $\sigma(ST) \subset \mathbf{R}^+ \cup \{0\}$

b) Suppose $0 \le S \le T$ and $ST = TS$. Show that $0 \le S^2 \le T^2$.

4) Let $H = L^2[0,1]$ with the inner product $\langle f, g\rangle = \int_0^1 f(t)\overline{g(t)}\,dt$, and let $Tf(t) = \int_0^t f(s)\,ds$.

a) What is $\|T\|$?

b) Show that there is a complete orthonormal basis $\{x_n\}_n$ of H such that $\sum_k \|Tx_n\|^2 < \infty$.

c) Show that

$$T^n f(t) = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s)\,ds$$

d) Using the method of Exercise 7.12[1] solve the equation

[1] Young, page 75

$$f(t) = \sin(t) + Tf(t)$$

5) Young, Exercise 7.41, p. 87.

March 24, 2010