1. Construct a martingale \((X_n)_{n \in \mathbb{N}}\) (on a suitable probability space, with respect to a suitable filtration) such that \((X_n)_{n \in \mathbb{N}}\) is not uniformly integrable but \(\sup_{n \in \mathbb{N}} E(|X_n|) < \infty\).

2. Let \((\mathcal{F}_t)_{t \in \mathbb{R}^+}\) be a filtration; show that the filtration \((\mathcal{F}_t^+)_{t \in \mathbb{R}^+}\) is right-continuous.

3. Let \(S, T\) be \((\mathcal{F}_t)_{t \in \mathbb{R}^+}\)-stopping times. Show that
   (a) \(T\) is \(\mathcal{F}_T\)-measurable.
   (b) \(S \leq T \Rightarrow \mathcal{F}_S \subset \mathcal{F}_T\).
   (c) \(S \wedge T, S \vee T\) are \((\mathcal{F}_t)_{t \in \mathbb{R}^+}\)-stopping times.
   (d) \(\mathcal{F}_{S \wedge T} = \mathcal{F}_S \cap \mathcal{F}_T\).
   (e) \(\forall A \in \mathcal{F}_S : A \cap \{S \leq T\} \in \mathcal{F}_T\).

4. Let \((S_n)_{n \in \mathbb{N}}\) be a sequence of \((\mathcal{F}_t)_{t \in \mathbb{R}^+}\)-stopping times.
   (a) Assume \((S_n)_{n}\) is a monotonically increasing sequence. Show that \(\lim_{n \to \infty} S_n\) is an \((\mathcal{F}_t)_{t \in \mathbb{R}^+}\)-stopping time.
   (b) Assume \((S_n)_{n}\) is a monotonically decreasing sequence. Show that \(\lim_{n \to \infty} S_n\) is an \((\mathcal{F}_t)_{t \in \mathbb{R}^+}\)-stopping time.

5. Let \(X = (X_t)_{t \in \mathbb{T}}\) and \(Y = (Y_t)_{t \in \mathbb{T}}\) be \((\mathcal{F}_t)_{t \in \mathbb{T}}\)-supermartingales (with \(\mathbb{T} \subseteq \mathbb{R}^+\)).
   (a) Show that \(X \wedge Y = (X_t \wedge Y_t)_{t \in \mathbb{T}}\) is an \((\mathcal{F}_t)_{t \in \mathbb{T}}\)-supermartingale.
   (b) Show that if \(X_\infty \in L^1(\Omega, \mathcal{F}, P)\) closes \(X\) to the right (i.e. \(E(X_\infty | \mathcal{F}_t) \leq X_t\) a.s. \(\forall t \in \mathbb{T}\)) and \(Y_\infty \in L^1(\Omega, \mathcal{F}, P)\) closes \(Y\) to the right, then \(X_\infty \wedge Y_\infty\) closes \(X \wedge Y\) to the right.

6. Let \(Y\) be a random variable, and let \(T\) be an \((\mathcal{F}_t)_{t \in \mathbb{R}^+}\)-stopping time. Show that \(Y\) is \(\mathcal{F}_T\)-measurable if \(Y1_{\{T \leq t\}}\) is \(\mathcal{F}_t\)-measurable \(\forall t \in \mathbb{R}^+\).

7. Let \(X = (X_t)_{t \in \mathbb{R}^+}\) be progressively measurable, and let \(T\) be an \((\mathcal{F}_t)_{t \in \mathbb{R}^+}\)-stopping time. Show that the stopped process \(X_T\) is progressively measurable.

8. Let \(A \subset \mathbb{R}^+ \times \Omega\); we define, for all \(\omega \in \Omega\), \(A^\omega = \{t \in \mathbb{R}^+ | (t, \omega) \in A\}\); the function \(D_A : \Omega \to \mathbb{R}^+ \cup \{\infty\}\) defined by \(D_A(\omega) = \inf A^\omega\) for all \(\omega \in \Omega\) is called the début of \(A\). Show that if the process \(X = (X_t)_{t \in \mathbb{R}^+}\) is right-continuous and adapted, and if the filtration \((\mathcal{F}_t)_{t \in \mathbb{R}^+}\) is right-continuous, then, for any open subset \(V\) of \(\mathbb{R}\), the début of the set \(\{X \in V\}\) is a stopping time. Does the conclusion hold if the filtration is not assumed right-continuous? if \(X\) is assumed only left-continuous?

9. On a suitable probability space \((\Omega, \mathcal{F}, P)\), equipped with a suitable filtration \((\mathcal{F}_t)_{t \in \mathbb{T}}\), construct a process \(X\) and a stopping time \(T\) such that \(X_T\) is not a random variable.