1. Let $(\mathcal{F}_n)_{n \in \mathbb{N}}$ be a filtration on $(\Omega, \mathcal{F}, P)$.
   
   (a) Let $X = (X_n)_{n \in \mathbb{N}}$ be an $(\mathcal{F}_n)_{n \in \mathbb{N}}$-martingale, right-closed by $X_\infty$ (i.e. $X_n = E(X_\infty|\mathcal{F}_n)$ a.s. $\forall n \in \mathbb{N}$).
      
      i. Show that for any stopping time $S$ (bounded or not), we have $X_S \in L^1$ (Hint: For arbitrary $k \in \mathbb{N}$, apply Doob’s Optional Stopping Theorem to the bounded stopping time $S \wedge k$, followed by Jensen and Fatou).
      
      ii. Show that for any stopping time $S$ (bounded or not), we have a.s. $E(X_\infty|\mathcal{F}_S) = X_S$.
      
      iii. Deduce from the above that for any two (not necessarily bounded) stopping times $S$ and $T$ with $S \leq T$, we have a.s. $E(X_T|\mathcal{F}_S) = X_S$.
   
   (b) Let now $X = (X_n)_{n \in \mathbb{N}}$ be a $(\mathcal{F}_n)_{n \in \mathbb{N}}$-supermartingale with $X_n \geq 0, \forall n \in \mathbb{N}$.
      
      i. Show that for any stopping time $S$ (bounded or not), we have $X_S \in L^1$ (Hint: Apply Doob’s Optional Stopping Theorem to the bounded stopping time $S \wedge k$, where $k \in \mathbb{N}$ is arbitrary, then use Fatou).
      
      ii. Deduce from the above that for any two (not necessarily bounded) stopping times $S$ and $T$ with $S \leq T$, we have a.s. $E(X_T|\mathcal{F}_S) \leq X_S$.
   
   (c) Let now $X = (X_n)_{n \in \mathbb{N}}$ be a $(\mathcal{F}_n)_{n \in \mathbb{N}}$-supermartingale right-closed by $X_\infty$ (i.e. $X_n \geq E(X_\infty|\mathcal{F}_n)$ a.s., $\forall n \in \mathbb{N}$). Using the decomposition $X_n = E(X_\infty|\mathcal{F}_n) + [X_n - E(X_\infty|\mathcal{F}_n)]$ of $X$ as the sum of a right-closed martingale and a positive ($\geq 0$) supermartingale, show that:
      
      i. For any stopping time $S$ (bounded or not), we have $X_S \in L^1$;
      
      ii. For any two (not necessarily bounded) stopping times $S$ and $T$ with $S \leq T$, we have a.s. $E(X_T|\mathcal{F}_S) \leq X_S$.

2. Let $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ be a filtration on $(\Omega, \mathcal{F}, P)$, let $S$ be a stopping time, and let $D_n = \{k2^{-n}\}_{k \in \mathbb{N}}, \forall n \in \mathbb{N}$. For each $n \in \mathbb{N}$ and $\omega \in \Omega$, let $S_n(\omega) = \inf\{t \in D_n \mid t \geq S(\omega)\}$. Show that for each $n \in \mathbb{N}$, $S_n$ is a $(\mathcal{F}_t)_{t \in D_n}$-stopping time.

3. Let $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ be a filtration on $(\Omega, \mathcal{F}, P)$, and let $T$ be a stopping time.
   
   (a) For each $n \in \mathbb{N}$, let $T_n = 2^{-n}[2^n T + 1]$ (where $[x]$ denotes the integer part of $x \in \mathbb{R}$); show that $T_n$ is a stopping time $\forall n \in \mathbb{N}$.
      
   (b) Show that $\forall n \in \mathbb{N}$, we have: $[0, T_n] = \{(0) \times \Omega \} \cup (\bigcup_{k \in \mathbb{N}}[k2^{-n}, (k + 1)2^{-n}] \times \{T \geq k2^{-n}\}$.

4. Let Let $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ be a filtration on $(\Omega, \mathcal{F}, P)$, and let $\mathcal{P}$ and $\mathcal{O}$ denote the predictable and optional $\sigma$–algebras, respectively.
   
   (a) Show that $\forall A \in \mathcal{P}, (1_A(t, \cdot))_{t \in \mathbb{R}_+}$ is $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$–adapted.
      
   (b) Show that $\forall A \in \mathcal{O}$, $(1_A(t, \cdot))_{t \in \mathbb{R}_+}$ is $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$–adapted.