1. A $\mathbb{R}$-valued stochastic process $(B_t)_{t \in \mathbb{R}^+}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is a Brownian motion (starting at 0) if
   (a) $B_0(\omega) = 0$, $\forall \omega \in \Omega$,
   (b) $t \mapsto B_t(\omega)$ is a continuous mapping on $\mathbb{R}^+$, $\forall \omega \in \Omega$,
   (c) $\forall s, t \in \mathbb{R}^+$ with $s \leq t$, $B_t - B_s$ is independent of $\sigma((B_r)_{0 \leq r \leq s})$.
   (d) $\forall s, t \in \mathbb{R}^+$ with $s \leq t$, $B_t - B_s$ has Gaussian distribution with mean 0 and variance $t - s$.

With $(B_t)_{t \in \mathbb{R}^+}$ denoting a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$, and with $\mathcal{F}_t = \sigma((B_r)_{0 \leq r \leq t}) \forall t \in \mathbb{R}^+$ (the natural filtration of the Brownian motion), show that the following are $(\mathcal{F}_t)_{t \in \mathbb{R}^+}$ martingales:
   (i) $(B_t)_{t \in \mathbb{R}^+}$,
   (ii) $(B_t^2 - t)_{t \in \mathbb{R}^+}$,
   (iii) $(e^{\alpha B_t - \frac{\alpha^2 t}{2}})_{t \in \mathbb{R}^+}$ (where $\alpha \in \mathbb{R}$).

2. Let $(\mathcal{F}_n)_{n \in \mathbb{N}}$ be a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$.
   (a) Let $X = (X_n)_{n \in \mathbb{N}}$ be an $(\mathcal{F}_n)_{n \in \mathbb{N}}$-martingale, right-closed by $X_\infty$ (i.e. $X_n = E(X_\infty|\mathcal{F}_n)$ a.s. $\forall n \in \mathbb{N}$).
      i. Show that for any stopping time $S$ (bounded or not), we have $X_S \in L^1$ (Hint: For arbitrary $k \in \mathbb{N}$, apply Doob’s Optional Stopping Theorem to the bounded stopping time $S \wedge k$, followed by Jensen and Fatou).
      ii. Show that for any stopping time $S$ (bounded or not), we have a.s. $E(X_\infty|\mathcal{F}_S) = X_S$.
      iii. Deduce from the above that for any two (not necessarily bounded) stopping times $S$ and $T$ with $S \leq T$, we have a.s. $E(X_T|\mathcal{F}_S) \leq X_S$.
   (b) Let now $X = (X_n)_{n \in \mathbb{N}}$ be a $(\mathcal{F}_n)_{n \in \mathbb{N}}$-supermartingale with $X_n \geq 0$, $\forall n \in \mathbb{N}$.
      i. Show that for any stopping time $S$ (bounded or not), we have $X_S \in L^1$ (Hint: Apply Doob’s Optional Stopping Theorem to the bounded stopping time $S \wedge k$, where $k \in \mathbb{N}$ is arbitrary, then use Fatou).
      ii. Deduce from the above that for any two (not necessarily bounded) stopping times $S$ and $T$ with $S \leq T$, we have a.s. $E(X_T|\mathcal{F}_S) \leq X_S$.
   (c) Let now $X = (X_n)_{n \in \mathbb{N}}$ be a $(\mathcal{F}_n)_{n \in \mathbb{N}}$-supermartingale right-closed by $X_\infty$ (i.e. $X_n \geq E(X_\infty|\mathcal{F}_n)$ a.s., $\forall n \in \mathbb{N}$). Using the decomposition $X_n = E(X_\infty|\mathcal{F}_n) + [X_n - E(X_\infty|\mathcal{F}_n)]$ of $X$ as the sum of a right-closed martingale and a positive ($\geq 0$) supermartingale, show that:
      i. For any stopping time $S$ (bounded or not), we have $X_S \in L^1$;
      ii. For any two (not necessarily bounded) stopping times $S$ and $T$ with $S \leq T$, we have a.s. $E(X_T|\mathcal{F}_S) \leq X_S$.

3. Let $(\mathcal{F}_t)_{t \in \mathbb{R}^+}$ be a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$, let $S$ be a stopping time, and let $D_n = \{k2^{-n}\}_k \in \mathbb{N}$, $\forall n \in \mathbb{N}$. For each $n \in \mathbb{N}$ and $\omega \in \Omega$, let $S_n(\omega) = \inf\{t \in D_n \mid t \geq S(\omega)\}$. Show that for each $n \in \mathbb{N}$, $S_n$ is a $(\mathcal{F}_t)_{t \in D_n}$-stopping time.

4. Let $X = (X_t)_{t \in \mathbb{R}^+}$ be a $(\mathcal{F}_t)_{t \in \mathbb{R}^+}$ $L^p$ martingale, with $1 < p < \infty$. Show that if $X$ is $L^p$-bounded (i.e. $\sup_t E(|X_t|^p) < \infty$), then $X$ is uniformly integrable.

5. Let $X = (X_n)_{n \in \mathbb{N}}$ be an $(\mathcal{F}_n)_{n \in \mathbb{N}}$ martingale. Assume $X_n \to X$ a.s. (for some $X \in L^1$), as $n \to \infty$. Show that if $E(|X_n|) \to E(|X|)$ as $n \to \infty$, then $X_n \to X$ in $L^1$. (Note: This result can be generalized to $L^p$, $1 \leq p < \infty$).