1. Let \((X_n)_{n \in \mathbb{N}}\) be a \((F_n)_{n \in \mathbb{N}}\)-local martingale, with \(X_n \geq 0, \forall n \in \mathbb{N}\) and \(X_0 = 0\). Show that \((X_n)_{n \in \mathbb{N}}\) is a \((F_n)_{n \in \mathbb{N}}\)-supermartingale.

2. Let \(S,T\) be stopping times with finite range (i.e. \(S(\Omega), T(\Omega)\) have finite cardinality). Show that \([S,T] \in \mathcal{E}\) (recall that \(\mathcal{E}\) is the vector subspace of \(\mathcal{P}\)-measurable processes which are linear combinations of indicator functions of predictable rectangles.)

3. Let \(A \subset \mathbb{R}^+ \times \Omega\); we define, for all \(\omega \in \Omega\),
   \[A_\omega = \{ t \in \mathbb{R}^+ \mid (t, \omega) \in A \}\];
   the function \(D_A : \Omega \to \mathbb{R}^+ \cup \{\infty\}\) defined by \(D_A(\omega) = \inf A_\omega\) for all \(\omega \in \Omega\) is called the début of \(A\). Show that if the process \(X = (X_t)_{t \in \mathbb{R}^+}\) is right-continuous and adapted, and if the filtration \((F_t)_{t \in \mathbb{R}^+}\) is right-continuous, then the début of the set \(\{X \in V\}\) is a stopping time, for any open subset \(V\) of \(\mathbb{R}\).

4. Let \((F_t)_{t \in \mathbb{R}^+}\) be a filtration on \((\Omega, \mathcal{F}, P)\), and let \(T\) be a stopping time.
   (a) For each \(n \in \mathbb{N}\), let \(T_n = 2^{-n} \lfloor 2^n T + 1 \rfloor\) (where \(\lfloor x \rfloor\) denotes the integer part of \(x \in \mathbb{R}\)); show that \(T_n\) is a stopping time \(\forall n \in \mathbb{N}\).
   (b) Show that \(\forall n \in \mathbb{N}\), we have: \([0, T_n] = ([0] \times \Omega) \cup \left( \bigcup_{k \in \mathbb{N}} \left(k 2^{-n}, (k + 1)2^{-n}\right] \times \{T \geq k 2^{-n}\} \right)\).

5. Let \((F_t)_{t \in \mathbb{R}^+}\) be a filtration on \((\Omega, \mathcal{F}, P)\), and let \(\mathcal{P}\) and \(\mathcal{O}\) denote the predictable and optional \(\sigma\)-algebras, respectively.
   (a) Show that \(\forall A \in \mathcal{P}\), \((1_A(t, \cdot))_{t \in \mathbb{R}^+}\) is \((F_t)_{t \in \mathbb{R}^+}\)-adapted.
   (b) Show that \(\forall A \in \mathcal{O}\), \((1_A(t, \cdot))_{t \in \mathbb{R}^+}\) is \((F_t)_{t \in \mathbb{R}^+}\)-adapted.

6. Let \(s < t \in \mathbb{R}^+\), and random variables \(X \in \mathcal{P}\), \(Z \in \mathcal{F}\), with \(X > 0\). Show that \(Z \in \mathcal{F}_s\) iff \(1_{[s,t]} Z X \in \mathcal{P}\).

7. Let \(X = (X_t)_{t \in \mathbb{R}^+}\) be a left-continuous adapted process, and \(\forall n \in \mathbb{N}\), define \((X^n)_{t \in \mathbb{R}^+}\) by
   \[X^n_t = \sum_{k \in \mathbb{N}} X_{k 2^{-n} 1_{k 2^{-n} \leq t < (k + 1)2^{-n}}} = \sum_{k \in \mathbb{N}} X_{k 2^{-n} 1_{k 2^{-n} \leq t < (k + 1)2^{-n}}} (\cdot)\).
   (a) Show that \(X^n\) is a predictable process \(\forall n \in \mathbb{N}\).
   (b) Deduce from the preceding that every left-continuous (and a fortiori continuous) process is predictable.